Choice Deferral, Indecisiveness and Preference for Flexibility

Leonardo Pejsachowicz and Séverine Toussaert*

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Abstract

We introduce a model of menu choice in which a person’s decisions may only partially reveal her innate tastes. The latter are modeled by means of a possibly incomplete (but otherwise rational) preference relation ≽, and the former by a completion ≽* of that relation. The two are connected through an axiom formalizing an intuitive rule: “Whenever in doubt, don’t commit; just leave options open.” Under the usual assumptions of the menu choice literature, we find that even the smallest amount of indecisiveness is enough to force ≽*, through this deferral property, to exhibit preference for flexibility on its entire domain. Thus we highlight a fundamental tension between non-monotonic preferences, such as preferences for self-control, and tendency to defer due to indecisiveness.

KEYWORDS: Incomplete preferences, preference for flexibility, choice deferral.

*Pejsachowicz: Department of Economics, Ecole Polytechnique, Route De Saclay, 91128 Palaiseau, France, Email: leonardo.pejsachowicz@polytechnique.edu. Toussaert: Department of Economics, New York University, 19 West 4th Street, New York, NY, Email: st1445@nyu.edu. For guidance, support and encouragement, we are extremely grateful to Efe Ok. We thank Eric Danan, Eddie Dekel, Faruk Gul, Elliot Lipnowski, Pietro Ortoleva, Andrei Savotchkin, John Stovall. We would also like to thank seminar participants at the Decision Theory Workshop and Micro Student Seminar at NYU. Leonardo Pejsachowicz acknowledges the support by a public grant overseen by the French National Research Agency (ANR) as part of the Investissements d’Avenir program (Idex Grant Agreement No. ANR-11- IDEX-0003-02 / Labex ECODEC No. ANR -11-LABEX-0047). Needless to say, all mistakes are our own.
1 Introduction

The primitive of the theory of choice among opportunity sets is a preference relation defined on a collection \(X\) of subsets of a given space of alternatives. These subsets, called “menus,” are generally interpreted as feasible sets from which an alternative will be selected at some later (unmodeled) stage. With this dynamic interpretation in mind, Kreps [38] introduced in a seminal paper a behavioral property called “preference for flexibility,” which is now a fairly common postulate in this literature.\(^1\) According to this postulate, the preference relation of the decision maker ranks any superset of a menu at least as high as that menu. Clearly, this monotonicity property seems particularly meaningful if the decision maker aims to accommodate unforeseen contingencies.\(^2\) However, it is by no means unexceptionable. There are situations in which one may prefer smaller menus to larger ones. Indeed, an agent may well favor commitment if he fears to be tempted by some option as in the model of Gul and Pesendorfer [32], or might prefer to restrict his choice set if he anticipates regret as in the model of Sarver [53].\(^3\) It seems therefore important to investigate the extent to which both of these concerns may be accounted for within a single framework and understand the circumstances under which one may be favored over the other.

In this paper, we start from the observation that a natural situation in which a decision maker may prefer to retain the flexibility of choice is when he is unable to decide between two courses of action. Our main result shows that this intuitive rule, which connects preference for flexibility to indecisiveness, may itself preclude the expression of any desire for commitment,

\(^1\) For instance, Nehring [42] studies preference for flexibility in a Savagean context. Dekel et al. [14, 15] extend the work of Kreps [38] to a lottery framework. Recently, Ahn and Sarver [1] combined preference for flexibility ex ante with random choice ex post. This property has also been recently analyzed in a dynamic setting (see, for example, Krishna and Sadowski [39]).

\(^2\) One common justification for preference for flexibility stems from the idea that the decision maker may feel uncertain about what his future tastes will be, and indeed, the representation theorems of Kreps [38] and Dekel et al. [14] give substance to this interpretation.

\(^3\) There is indeed a vast literature on non-monotonic preferences over menus. These models have been used to capture a variety of phenomena, from perfectionism (Kopylov [37]), to bias in belief updating (Epstein et al. [22]), to shame (Dillenberger and Sadowsky [17]), to thinking aversion (Ortoleva [49]), to anxiety from risk (Epstein [21]). A comprehensive review can be found in Lipman and Pesendorfer [41].
thus leaving preference for flexibility as the sole concern of the agent.

The idea of indecisiveness is of course not new in decision theory; it
dates back to Aumann [3], and is captured by dropping the assumption of completeness of one’s preferences. Moreover, while it is mostly studied
in other contexts, indecisiveness is likely to be more pronounced in the
context of menu preferences. After all, one of the main justifications for
allowing for indecisiveness stems from the complex nature of the objects
of choice (such as goods with multiple attributes or lotteries with a large
support), which the decision maker may find hard to compare. This source of
indecisiveness was originally noted by Aumann [3] who wrote that some “…
decision problems might be extremely complex, too complex for intuitive
“insight”, and our individual might prefer to make no decision at all in
these problems.” In this respect, one would expect the situation to be no
easier when the objects of choice are themselves decision problems, as is
the case in the menu choice framework. It thus appears that representing
the psychological tastes of an agent by a potentially incomplete preference
relation on X is quite natural in the present setting. By A ≽ B, we
understand that there is no doubt in the mind of the agent that the menu
A is better than B. Nevertheless, we leave open the possibility that neither
A ≽ B nor B ≽ A may hold.

Yet, more often than not, we do not observe the inherent tastes of a
decision maker; we rather see the choices he makes. Suppose that from the
feasible set \{A, B\} of menus, we see him choosing A instead of B. We then
say that A is revealed preferred to B by this agent. Assuming that a choice
has to be made at all times, this gives rise to a complete preference relation,
say, ≽∗, on X. If A ≽∗ B, we understand that the agent declared A better
than B through his choice, but we do not know if A ≽ B actually holds.
Perhaps the agent was unable to compare A and B on the basis of his core
preferences, and his choice of A over B followed from the recommendation

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4There is now a sizable literature on the theory of incomplete preferences. The recent
contributions can be found on a large spectrum, ranging from the ordinal framework of
Peleg [50], Ok [46] or Evren and Ok [26] to the cardinal world of lotteries (Dubra et al.
[18] and Baucells and Shapley [4]) or choice under uncertainty (Bewley [5], Ok et al. [48],
Galaabaatar and Karni [28], Riella [52]).

5By preference relation we always mean here a reflexive and transitive binary relation.

6This might be the case if for instance he cannot foresee how he will feel in the future,
when choosing from the selected menu. Alternatively, his tastes might be so genuinely
incomplete that he cannot make up his mind between some sets even when they contain
a single alternative. The idea that indecisiveness may be understood either as incom-
pleteness in beliefs (here about one’s tastes tomorrow) or as incompleteness in tastes was
recently formalized by Ok et al. [48] in the context of decision making under uncertainty.
of a second party and/or from the adoption of some ad hoc choice procedure. By contrast, if \( A \succ B \) actually holds, we surely expect \( A \succ^* B \) to hold.

The literature on menu preferences usually takes the revealed (complete) preference relation \( \succ^* \) on \( X \) as the primitive of the analysis. By contrast, in this paper, we take two preference relations \( \succ \) and \( \succ^* \) on \( X \) as the primitives. As discussed above, \( \succ \) is possibly incomplete and corresponds to the innate tastes of the agent, while \( \succ^* \) is a completion of \( \succ \) and captures the behavior of the agent. Thus, when \( \succ = \succ^* \), our setting reduces to the standard framework.\(^7\)

This modeling approach allows us to explore the connections between an agent’s indecisiveness and his preference for flexibility. On the one hand, indecisiveness is a psychological phenomenon, which is captured by the potential incompleteness of the core relation \( \succ \). On the other hand, preference for flexibility is a behavioral phenomenon, which is captured by the monotonicity of \( \succ^* \) (which formally means that \( A \succ^* B \) for any two menus \( A \) and \( B \) with \( A \supseteq B \)). Our main axiom builds a bridge between those two phenomena and allows us to investigate the implications of indecisiveness (of \( \succ \)) for monotonicity (of \( \succ^* \)). This axiom formalizes the following intuitive rule: “Whenever in doubt, don’t commit; just leave options open.” At a more formal level, this rule says that if \( A \) and \( B \) are incomparable according to \( \succ \), one would expect that \( A \cup B \succ^* A \) (but we do not know whether \( A \succ^* B \) or \( B \succ^* A \)). Intuitively, an indecisive decision maker will often seek to defer choice if, for instance, he expects to be better informed in the future or simply needs additional time for contemplation about a difficult decision. Under such circumstances, choosing not to commit to a given menu can be seen as a cautious attitude. We thus call this property the **Cautious Deferral Axiom.**\(^8\)

Our primary interest is to understand the behavioral consequences of the Cautious Deferral Axiom. In order to do so, we adopt the standard frame-

\(^7\)Models in which the individual is endowed with two relations have been the object of a recent investigation. Similarly to our work, Gilboa et al. [31] and Kopylov [36] in the Anscombe-Aumann setup, and Danan [12] in a general menu choice setting, take as their primitive an incomplete relation and its completion. Close to this approach is the work by Lehrer and Teper [40], who consider an incomplete relation and its extension, without the further requirement that the latter be complete. Nehring [43] focuses on a complete relation on acts and an incomplete likelihood on events. The element common to all these works is therefore the presence of two primitive relations.

\(^8\)Note that this property is specific to the language of menus, which allows the formulation of the notion of “choice deferral” through the agent’s preference for flexibility. For this reason, such a connection between a preference relation and its completion has not been explored in decision theory. One exception is Danan [12], on which more shortly.
work of the theory of menu preferences and impose the usual rationality assumptions of the literature, namely Continuity and Independence. Our main result (Section 2.4) highlights a connection between Cautious Deferral and monotonicity. When the agent faces no internal conflict, that is, when $≽$ is complete, our rule imposes no restriction on choice behavior. On the other hand, we find that even a minimal amount of indecisiveness has far-reaching consequences: provided that $≽$ is not complete, $≽^*$ must exhibit preference for flexibility on its entire domain and this, regardless of the extent of the incompleteness of $≽$. By building a bridge between indecisiveness and preference for flexibility, the Cautious Deferral Axiom therefore gives credence to the proverb “Indecision is the key to flexibility”.

One can also view our main result as an impossibility theorem (Section 2.5). To wit, consider for instance an agent with incomplete tastes who satisfies the basic rationality postulates but may violate set monotonicity because he suffers from temptation à la Gul and Pesendorfer [32]. Then our main theorem shows that if this agent also abides by the Cautious Deferral rule, temptation, self-control and flexibility motives all disappear. What we are left with is a standard decision maker who evaluates a set by its maximal elements with respect to the expectation of some utility function. Thus, in the usual menu choice framework, we identify a basic tension between non-monotonic preferences, indecisiveness and the Cautious Deferral Axiom.

A third but equally compelling interpretation of our work is as an indirect criticism, and as a new showcase, of the power of the Independence assumption. The linear structure that this axiom induces in our model is in fact essential to our result, since it allows the consequences of the local interaction between indecisiveness and the Cautious Deferral Axiom to spread globally (see discussion in Section 3.2). Thus, one might argue that the real cause of the short circuit we observe between deferral, indecisiveness and preference for commitment is the tight set of constraints that Independence imposes on behavior. While our paper is not the first one to take issue with this assumption, we believe that the connection we highlight sheds new light on the discussion.

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9For example, Epstein et al. [23] propose that a decision maker with a coarse perception of future contingencies may have hedging motives leading him to only satisfy an adaptation of the Uncertainty Aversion Axiom of Gilboa and Schmeidler [30] to the menu choice setting. At the opposite extreme, Ergin and Sarver [24] argue that an agent for whom contemplation is costly will be averse to contingent planning, implying a preference for making a choice only after uncertainty is resolved. In the temptation literature, Noor and Takeoka [44] and [45] propose models à la Gul and Pesendorfer in which the cost of resisting temptation is either convex or menu dependent, which necessitates a violation of the affinity assumption.
2 The Model

2.1 Preliminaries

We work in the standard framework of the theory of menu preferences as in, say, Dekel et al. [14] (from now on DLR). In what follows, we let $\Delta$ stand for the set of all probability distributions (lotteries) over a finite prize space $Z$ with $|Z| = n$. The generic members of $\Delta$ are denoted as $a, b, c, \ldots$ etc. We let $X$ stand for the set of all nonempty closed subsets of $\Delta$. As is standard in this literature, we view $X$ as a metric space relative to the Hausdorff metric. The generic members of $X$ are denoted as $A, B, C, \ldots$ etc., and are referred to as menus. These sets are interpreted as opportunity sets from which the decision maker will choose an option at a later (unmodeled) stage.

In what follows, we also consider $X$ as an algebraic entity by imposing on it the mixture operation induced by the Minkowski sum of sets. That is, for any $A$ and $B$ in $X$ and any $0 \leq \lambda \leq 1$, we define

$$\lambda A \oplus (1-\lambda)B := \{\lambda a + (1-\lambda)b \mid a \in A \text{ and } b \in B\}$$

which is itself an element of $X$.

A binary relation $\succeq$ on $X$ is said to be a preorder, or a preference relation, on $X$ if it is reflexive and transitive. The asymmetric part of this relation is denoted by $\succ$, that is, we have $A \succ B$ if and only if $A \succeq B$ but not $B \succeq A$. (The symmetric part of $\succeq$, denoted by $\sim$, is thus $\succeq \setminus \succ$.) A preorder $\succeq$ on $X$ is said to be monotonic if $A \succeq B$ holds for every $A$ and $B$ in $X$ with $A \supseteq B$.

We denote the non-comparability part of a preorder $\succeq$ by $\nsucc$. That is, $\nsucc$ is the binary relation on $X$ such that $A \nsucc B$ if and only if neither $A \succeq B$ nor $B \succeq A$. If $\nsucc = \emptyset$, then $\succeq$ is said to be complete. In turn, given a preference relation $\succeq$ on $X$, by a proper completion of $\succeq$, we mean a complete preference relation $\succeq^*$ such that

$$A \succeq B \text{ implies } A \succeq^* B \quad \text{and} \quad A \nsucc B \text{ implies } A \nsucc^* B$$

We say that an ordered pair $(\succeq, \succeq^*)$ is a preference structure on $X$ if $\succeq$ is a preference relation on $X$ and $\succeq^*$ is a proper completion of $\succeq$.\(^{10}\) In this paper, we will consider such structures as the primitives of the environment.\(^{11}\) Of course, when $\succeq = \succeq^*$, the induced preference structure can be

\(^{10}\)A weaker version of this assumption without the strict part of the requirement, is present as an axiom in Gilboa et al. [31] under the name of Consistency.

\(^{11}\)We note that such preference structures have been previously studied by Danan [12], a discussion of which is provided in Section 4.
identified with a complete preference relation, so the standard framework of menu preferences is a special case of ours.

We interpret a preference structure \((\succeq, \succ^*)\) as follows. The first relation \(\succeq\) represents the decision maker’s psychological tastes. Simply put, this core relation captures among which menus the agent is completely decisive. As such, \(\succeq\) is unobservable to the modeler. On the other hand, the second relation \(\succ^*\) stems from the revealed preferences of the agent through his choice behavior. As we presume that we observe all choices of the agent across pairwise problems, this relation is taken as complete. Furthermore, it is consistent with the core relation of the agent, that is, whenever \(\succeq\) ranks menus in a particular manner, \(\succ^*\) does so in exactly the same way.

2.2 Basic Axioms

Throughout the paper, we will work with a preference structure \((\succeq, \succ^*)\) on \(X\) which is rational in the standard sense. That is, we will impose the following two axioms on \((\succeq, \succ^*)\):

**Axiom 1** (Independence): For every \(A, B, C \in X\) and \(0 < \lambda < 1\), we have

\[ A \succeq B \text{ if and only if } \lambda A \oplus (1 - \lambda) C \succ \lambda B \oplus (1 - \lambda) C \]

and similarly for \(\succ^*\).

**Axiom 2** (Continuity): Both \(\succeq\) and \(\succ^*\) are closed in \(X \times X\).\(^{12}\)

Axiom 1 is an adaptation of the standard Independence Axiom to a menu choice setting, which was first introduced by DLR \([14]\) and Gul and Pesendorfer \([32]\) and is now widely used in this literature. The logic for this axiom is the following. First, notice that if an agent satisfies the classical Independence condition, then whenever she prefers \(A\) to \(B\), she must also prefer \(\lambda A + (1 - \lambda) C\) (i.e. the lottery that gives menu \(A\) with probability \(\lambda\) and menu \(C\) with probability \((1 - \lambda)\)) to \(\lambda B + (1 - \lambda) C\). Secondly, if the agent is indifferent as to when uncertainty is resolved, then he should be indifferent between the lottery \(\lambda A + (1 - \lambda) C\), which provides randomization before the agent has chosen an item from the menu, and the set mixture \(\lambda A \oplus (1 - \lambda) C\), where randomization takes place ex post. Axiom 1 therefore logically follows from the assumptions that the agent satisfies the standard

\(^{12}\)\(X \times X\) is here the product metric space with Hausdorff underlying metric on \(X\).
Independence Axiom and is indifferent as to the timing of the resolution of uncertainty.

Axiom 2, also known as Closed-Continuity, is a technical assumption, which is fairly standard in decision theory. For a complete preference relation it is equivalent, under appropriate topological conditions on the choice space, to continuity of the representing functional.\(^{13}\) It is important to highlight that, for an incomplete relation, Axiom 2 also implies that \(\sqsupset\) will be open. Thus in a certain sense incomparability, when present, will never be too “small”.\(^{14}\) This turns out to be essential for our result, since we extensively use the implied richness of the incomparability relation.\(^{15}\) We would like to note here that, even though continuity-like assumptions are by nature unfalsifiable, and thus a discussion of their relative value has no empirical relevance, we do in fact feel that requiring openness of \(\gg\) has descriptive appeal. It is very hard indeed to imagine that a decision maker would declare himself unable to decide between two alternatives \(a\) and \(b\) and yet be able to compare two very close substitutes of those options. For example, for a worker who is fundamentally undecided between two hardly comparable job offers, say one in the financial industry and one in the public sector, it is very hard to believe that changing minor aspects of the two contracts, such as the level of dental coverage or the number of sick leave days, would lead to a clear preference. In this sense, Axiom 2 captures the idea that incomparability is a “robust” relation.

2.3 Cautious Deferral

We now introduce our main axiom, called the Cautious Deferral Axiom, which connects the preference relations \(\succeq\) and \(\succeq^*\).\(^{16}\)

\(^{13}\)For complete relations, it is also equivalent to openness of the strict part \(\succ\). While it is well known since Eilenberg [19] and Debreu [12] that, for many topological spaces, this condition is sufficient for the existence of a utility representation, it is not always necessary. For example, the expected utility theory for preferences over lotteries relies on a weaker condition, Archimedean Continuity, which only restricts the Bernoulli function to being bounded over the prize space.

\(^{14}\)In particular, whenever \(A \gg B\), there will be two open neighbourhoods \(O_A\) of \(A\) and \(O_B\) of \(B\) such that \(C \gg D\) if \(C\) is in \(O_A\) and \(D\) is in \(O_B\). When the choice space is Euclidean, this implies that the set of options incomparable to an element is either empty or full dimensional.

\(^{15}\)For an examination of the different implications of closed and open continuity in the context of incomplete relations, see Evren and Ok [26] and Evren [25].

\(^{16}\)Although very different in their structure and context, several papers present similarities with the approach adopted here by considering a pair of preference relations connected through some behavioral axiom such as our Cautious Deferral rule. For instance in the
Axiom 3 (Cautious Deferral): For every $A$ and $B$ in $X$,

$$A \succ B \quad \text{implies} \quad A \cup B \succ^* A$$

Note that one can always view an incomplete preference $\succ$ as the intersection of a collection of complete preferences, each of which represents a different criterion of evaluation in the mind of the agent.\(^{17}\) Thus, the decision maker’s inability to compare two menus through $\succ$ can be seen as stemming from the conflict between the various considerations that may enter his evaluation of the problem, or equivalently, between his different “selves” where the tastes of each self are represented by a complete preference relation on $X$. For instance, consider an agent who must decide between two restaurants $A$ and $B$ and evaluates each menu according to how healthy and how appetizing the available options are in each menu. If menu $A$ contains the most healthy option (option $a$) and menu $B$ contains the most appetizing option (option $b$), the agent might be unable to decide at which restaurant to dine. This inability to compare two menus $A$ and $B$ is captured in our model by setting $A \succ B$.

When intimately torn between two menus $A$ and $B$, what rule of conduct may the decision maker adopt? If the agent is constrained to choose from the feasible set $\{A, B\}$, our model remains silent on what the final choice will be. All we know is that $A \succ^* B$ if this choice is $A$, and similarly $B \succ^* A$ if $B$ is selected. (Recall that $\succ^*$ stands for the revealed preference of the agent.) As $A \succ B$, we also do not have any rationale for why either of these choices might occur. (For instance, the agent may have consulted a second party who suggested $A \succ^* B$ or he may have adopted some ad hoc choice procedure.) But now suppose that his choice set is actually $\{A, B, A \cup B\}$. In the context of the example above, our agent may have the additional option of going to a restaurant with a larger menu where the options of both restaurants $A$ and $B$ are available. Alternatively, one might think of $A \cup B$ as waiting until dinner time in order to pick a restaurant rather than making a reservation at either $A$ or $B$. In either case, it stands to reason that the decision maker would seize this additional opportunity, which would mean $A \cup B \succeq^* A$ (and by symmetry, $A \cup B \succeq^* B$), since this leaves all

\[\text{Anscombe-Aumann framework, Gilboa et al. [31] connect two preference relations, one incomplete and one complete, with an axiom called Caution which captures the agent’s behavioral attitude towards uncertainty. That paper, as well as other papers which adopt such an approach, are discussed in Section 4.}\]

\(^{17}\)This is a set-theoretic fact; see Section 1.4 of Ok [47] for more details.
options open at the later stage: by not committing to either menu, deferral is a cautious attitude for the agent. This suggests that perhaps we have \( A \cup B \succ A \) and therefore, \( A \cup B \succ^* A \), meaning that the agent intimately prefers the flexibility offered by the wider menu. Or perhaps the agent is still indecisive between \( A \) and \( A \cup B \), that is, \( A \succ A \cup B \). Even in this case, it seems that committing today to \( A \) as opposed to \( A \cup B \), that is, \( A \succ^* A \cup B \), is unwarranted. For instance, suppose that \( \succ^* \) results from the advice of a second party. Then a natural recommendation of this party would be “if you are not sure what menu to commit to, choose not to commit at all and wait if this is possible”. This suggests again \( A \cup B \succ^* A \), which is exactly what the Cautious Deferral Axiom says.

This discussion points to the intuitive nature of the Cautious Deferral Axiom, but it does not say anything about why a conflicted decision maker may find choice deferral desirable. In our view, this may happen for at least two reasons. On the one hand, an indecisive agent might prefer to postpone his choice if he expects his internal conflict to be resolved in the future. For instance, the agent in our example might later find himself in an indulgent mood, thus making restaurant \( B \) a clearly more attractive choice than \( A \). In this respect, choice deferral may have an informational value to the decision maker. On the other hand, even when the agent does not expect his indecisiveness to resolve in time, choice deferral might be valued if it helps the decision maker to come to terms with a difficult decision. By leaving options open, the agent allows himself additional time for contemplation about this difficult choice problem.\(^{19}\)

Finally, we wish to emphasize that the Cautious Deferral Axiom should be interpreted as having a positive, rather than a normative, content. Indeed, introspection and anecdotal evidence seem to provide support for this intuitive rule.\(^{20}\) Furthermore, several empirical studies attest to the descriptions.

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\(^{18}\)To illustrate this possibility, consider an agent who has two “selves”. One has self-control preferences à la Gul and Pesendorfer [32] and assesses the value of a menu \( A \) as \( W_1(A) = \max_{a \in A} [u(a) + v(a)] - \max_{a \in A} v(a) \), while the other evaluates \( A \) as a weighted average of the “commitment” and “temptation” utilities, that is, \( W_2(A) = \lambda \max_{a \in A} u(a) + (1 - \lambda) \max_{a \in A} v(a) \), where \( \lambda \) is a real number in \( (0,1) \). Now assume that the agent needs to choose between \( \{a\} \) and \( \{a,b\} \), and that \( u(a) > u(b) \) while \( v(b) > v(a) \). Then, we have \( \{a\} \triangleright \{a,b\} \) because \( W_1(\{a\}) > W_1(\{a,b\}) \) and \( W_2(\{a,b\}) > W_2(\{a\}) \).

\(^{19}\)A dramatic illustration of this case can be found in the novel of William Styron, Sophie’s Choice. One can think that the choice Sophie faced between saving her son from the gas chamber or saving her daughter might have been anything but a spontaneous decision.

\(^{20}\)Tversky and Shafir [55] report an anecdote that was shared by Thomas Schelling. The latter had decided to buy an encyclopedia for his children but, to his discontent, he found two in the bookstore. Even though both options seemed satisfactory, the difficulty
tive appeal of this axiom by documenting a link between choice deferral and the inability to compare alternatives (Tversky and Shafir [55], Dhar [16], Tykocinski and Ruffle [56]; see Section 4 for more details). At the same time, we concede that in some contexts, indecisiveness might instead result in a desire for commitment, for instance if the decision maker suffers from temptation or cognitive load. We return to this issue in Section 2.5 where we discuss the tensions between commitment and flexibility.

2.4 Main Result

The main objective of the present work is to understand the implications of the Cautious Deferral Axiom with respect to the preference for flexibility that may be exhibited by the choice behavior of a decision maker. More precisely, we wish to understand the extent to which the behavioral preference relation $\succeq^*$ may exhibit a preference for flexibility when the preference structure $(\succeq, \succeq^*)$ on $X$ satisfies Axiom 3. It is plain that the answer depends on how incomplete the psychological preference relation $\succeq$ is. At one extreme of the spectrum is the case where this relation is complete, that is, $\succsim = \emptyset$. In this case, Axiom 3 becomes a triviality, yielding no clue as to the structure of $\succeq^*$. At the other extreme is the case where $\succeq$ is unable to rank any two distinct menus, that is, $\succsim = \{(A, B) : A \neq B\}$. In that case, it is readily verified that Axiom 3 implies the monotonicity of $\succeq^*$, that is, $A \succeq^* B$ for every $A$ and $B \in X$ such that $A \supseteq B$. Therefore, while the Cautious Deferral Axiom enforces some monotonicity on $\succeq^*$, the scope of this monotonicity, and in particular, whether or not $\succeq^*$ is monotonic on its entire domain, depends on the degree of indecisiveness exhibited by the core preference relation $\succsim$.

The main result of this paper shows that, in the context of rational preferences structures on $X$, the restrictions on $\succeq^*$ imposed by the Cautious Deferral Axiom are actually much stronger than the previous discussion suggests. Indeed, for preference structures $(\succeq, \succeq^*)$ that satisfy Axioms 1-2 as well as the Cautious Deferral Axiom, it turns out that the full completeness of $\succeq$ is the only way in which $\succeq^*$ may escape from exhibiting a global preference for flexibility. Even when the psychological preference relation $\succeq$ presents only a minimal amount of incompleteness, that is, whenever $\succsim \neq \emptyset$, it turns out that $\succeq^*$ is sure to be monotonic on its entire domain. Under the usual rationality postulates, the Cautious Deferral Axiom thus implies to make a choice between the two encyclopedias led Schelling to buy neither.
preference for flexibility so long as the core preferences $\succ$ of the individual are incomplete, and this, regardless of the extent of the incompleteness of $\succ$.

**Theorem 1**: Let $(\succ, \succ^*)$ be a preference structure on $X$ which satisfies Axioms 1-3. Then, either $\succ$ is complete or $\succ^*$ exhibits preference for flexibility.

On the one hand, this theorem shows that there is a fundamental tension between non-monotonic preferences over menus, incompleteness of one’s tastes and the Cautious Deferral Axiom. In particular, an agent with incomplete tastes cannot exert commitment on any part of the menu space while at the same time completing his core preferences in concert with the Cautious Deferral Axiom. On the other hand, this theorem appears to provide a psychological foundation for the property of preference for flexibility.

Insofar as one concedes that a rational decision maker may sometimes be indecisive and also that her revealed preferences are consistent with the Cautious Deferral Axiom, then those preferences must exhibit preference for flexibility in all contingencies. In accordance with the opening vignette of the paper, we thus find that indecisiveness (on the part of the psychological preferences $\succ$) is the key to preference for flexibility (on the part of the revealed preference $\succ^*$) for rational individuals who behave consistently with the Cautious Deferral Axiom.\(^{21}\)

One natural question at this point is if the core relation $\succ$ is also forced to be monotonic (when it is incomplete) under the conditions of Theorem 1. We show next by an example that this need not be the case.

**Example**: Consider the real maps $W_1$ and $W_2$ on $X$ defined by

$$W_1(A) := \lambda \max_{a \in A} w(a) - \max_{a \in A} v(a)$$

and

$$W_2(A) := \lambda \max_{a \in A} v(a) - \max_{a \in A} w(a),$$

where $w$ and $v$ are distinct, continuous and affine functions on $\triangle$ and $\lambda > 1$. Then, it can easily be seen that the relation $\succ$ defined as

$$A \succ B \text{ if and only if } W_1(A) \geq W_1(B) \text{ and } W_2(A) \geq W_2(B),$$

\(^{21}\)Here the reader may reasonably wonder whether more indecisiveness implies a stronger preference for flexibility in our model. In Section 2.5 of the Online Appendix (available on the authors’ website), we provide a counterexample to show that this is not the case. The reason is that the restrictions imposed by the Cautious Deferral Axiom are too weak to provide us with this kind of comparative statics.
is incomplete and satisfies Axioms 1 - 2. On the other hand, the preference relation $\succ^*$ on $X$ represented by $W_1 + W_2$ is a completion of $\succ$ which satisfies the Cautious Deferral Axiom. Therefore, in line with our main theorem, $\succ^*$ exhibits preference for flexibility but $\succ$ is not monotone on $X$.

2.5 Self-Control versus Cautious Deferral

One way of interpreting our main theorem is as a negative result, which shows that non-monotonic preferences must be renounced if one wishes to maintain indecisiveness together with Cautious Deferral. To demonstrate this point, consider the Gul and Pesendorfer [32] model of temptation and self-control. This model takes as a primitive the revealed preference relation $\succ$ on $X$, and besides the standard rationality assumptions, imposes the Set Betweenness Axiom. This axiom states that $A \succ B$ implies $A \succ A \cup B \succ B$, for any two menus $A$ and $B$. To understand the first part of the implication, consider an agent for whom $A \succ B$ because the best alternative in $A$ is strictly preferred to the best alternative in $B$. Then, the axiom allows for the agent to strictly prefer committing to $A$ today rather than to $A \cup B$, that is, $A \succ A \cup B$, which is rather reasonable if $B$ contains a temptation that the agent wishes to avoid.

Even without appealing to the representation theorem of Gul and Pesendorfer [32], we can show that the Cautious Deferral Axiom goes against the agent’s desire for commitment. To this end, suppose that the core preferences of the agent are represented by some incomplete preference relation $\succ$, and $\succ^*$ is a completion of $\succ$. Now take any two menus $A$ and $B$. If $A \succ B$, then the motivation behind the Set Betweenness Axiom applies, so we may comfortably posit that $A \succ A \cup B$. But suppose that $A \succ B$ is the case, that is, the agent is unable to decide (today) between $A$ and $B$ on the basis of his core preferences. Then, the Cautious Deferral Axiom tells us that the agent’s fear of making a mistake by committing to either $A$ or $B$ would overwhelm any other consideration, prompting him to set $A \cup B \succ A, B$. At an intuitive level, this behavior seems to make perfect sense: the agent exerts self-control when he can clearly identify the elements of temptation that may be present (i.e. $A \succ B$ implies $A \succ A \cup B$ and hence $A \succ A \cup B$), but wishes not to commit when comparing two menus between which he is indecisive (i.e. $A \succ B$ implies $A \cup B \succ A$).

This intuition runs, however, to a severe difficulty, at least in the presence of the standard rationality axioms. For, take any two menus $A$ and $B$ such that $A \succ B$. By the Set Betweenness Axiom, it must be that
$A \succ^* A \cup B$. If, however, $\succ^*$ is also a completion of $\succ$, and both relations satisfy all the standard axioms and the Cautious Deferral Axiom, then we know by Theorem 1 that $\succ^*$ must be monotonic on its entire domain, so that $A \cup B \succ^* A$. Therefore, $A \succ^* B$ implies $A \sim^* A \cup B$, that is, our agent must be a standard decision maker who evaluates a menu by its best elements. Put differently, self-control motives are completely annihilated in the presence of the Cautious Deferral Axiom.

One could reasonably argue that the flip side of Cautious Deferral, which would require that when $A \succ B$ then $A \succeq^* A \cup B$, might be a more appropriate assumption in the context of temptation. Indeed, if the decision maker feels indecisive between two menus because of the respective temptations they contain, then committing to either menu would seem to be a more cautious attitude than leaving options open: by avoiding exposure to one of the temptations, the agent can increase his chances of exercising self-control. We therefore coin this axiom Cautious Avoidance. Given the structure of our proof, it can be easily verified that a symmetric result would be obtained if Cautious Deferral were to be replaced by Cautious Avoidance, this time with the conclusion that nonempty incomparability implies $A \succ^* B$ whenever $A \subseteq B$, a property we dub global weak preference for commitment.\footnote{To see this in a concrete example, consider an agent who evaluates sets by combining the value of the best normative choice and that of the strongest temptation. Suppose he faces the options: broccoli $b$, potato chips $p$ and chocolate cake $c$. Potato chips and chocolate cake are more tempting than broccoli but the chips are more tempting as a salty craving, while the cake is more tempting as a sweet craving. On the other hand, broccoli is healthier than either of the options. If the agent is unable to determine which is stronger between the salty and the sweet craving, it will be the case that $\{b, p\} \succ \{b, c\}$. On the other hand, it makes sense that he would always choose either $\{b, p\}$ or $\{b, c\}$ over $\{b, c, p\}$, since the latter set guarantees the worst temptation. Such agent would clearly violate Cautious Deferral (we thank John Stovall for providing this example).}

**Axiom 4 (Cautious Avoidance):** For every $A$ and $B$ in $X$,

$$A \succ B \implies A \succeq^* A \cup B$$

**Theorem 2:** Let $(\succ, \succ^*)$ be a preference structure on $X$ which satisfies Axioms 1, 2 and 4. Then, either $\succ$ is complete or $\succ^*$ exhibits global weak preference for commitment.\footnote{Preference for commitment, as defined by Gul and Pesendorfer \cite{32}, requires that there exists at least a pair $A, B$ such that $A \subset B$ and $A \succ B$. Hence the qualifiers: global because in our case it applies to all sets, weak because a strict preference is not required.}
We can now consider what additional restrictions would appear if Cautious Avoidance were to be combined with Set Betweenness. Take any two menus $A$ and $B$ such that $A \succ^* B$. By the Set Betweenness Axiom, it must be that $A \succeq^* A \cup B \succ^* B$. On the other hand, under our rationality assumptions, Theorem 2 implies that $B \succeq^* A \cup B$. As a result, $A \succ^* B$ implies that $A \succ^* A \cup B \sim^* B$. In the language of Gul and Pesendorfer [32], $B$ is therefore an overwhelming temptation, meaning that the agent expects to exert no self-control when facing a temptation in $B$. Since this can be done for any pair $(A, B)$, the agent faces overwhelming temptation at every set.\(^{24}\)

### 3 Discussion

Our framework is parsimonious in that it solely relies on three axioms, two of which are generally considered as the basic assumptions for rational preferences. Yet, we obtain a rather strong result for the restrictions imposed by these axioms on our preference structure. In this section, we provide some intuition for the proof of our main theorem and discuss the role played by our rationality assumptions. We conclude by discussing the observable implications of our axiomatic framework.

#### 3.1 Proof Sketch

The proof relies on the observation that, if the completion $\succeq^*$ establishes that $A \succ^* B$ for some sets $A$ and $B$ such that $A \subset B$, then the Cautious Deferral Axiom must force $A \succ B$. In other words, the underlying psychological relation $\succ$ must be able to compare every pair $(A, B)$ for which the completion violates monotonicity. The proof builds upon this observation in order to show that the mere existence of a single pair $(A, B)$ at which $\succ^*$ violates monotonicity is sufficient to guarantee that $\succeq^*$ is complete.\(^{25}\)

While we leave the details of the proof to the appendix, we note that the idea is based on an adaptation of the classical constructive proof of the existence of a Von Neumann-Morgenstern utility. Starting with sets $A$, $B$ and...

\(^{24}\)It can be verified that the preferences of such a decision maker can be represented by the function $V : X \to \mathbb{R}$ given by $V(A) = -\max_{a \in A} v(a)$, where $v$ is an expected utility function.

\(^{25}\)Obviously, thanks to Independence of $\succ^*$, a violation of monotonicity at a pair $(A, B)$ implies such violations at many other points, a fact we return to in the next section (3.2) and in one of the counterexamples we consider in the Online Appendix. Also, notice that the above statement, that a violation of monotonicity at a single pair $(A, B)$ ensures completeness of $\succeq$, is the contrapositive of the statement of our theorem.
C such that $A \subset B \subset C$ and $A \succ^* C \succ^* B$, one can show that the Cautious Deferral Axiom entails that $C$ must be ranked by $\succeq$ between $A$ and $B$, and it can be assigned a unique utility value. After this first step, it can be shown that the whole space of lotteries $\triangle$ can be uniquely ranked and assigned a utility value. This step follows from the affinity implied by the Independence Axiom and because Continuity implies that $A \succ^* \alpha C \oplus (1-\alpha)\triangle \succ^* B$ for large enough $\alpha$. A third repetition of the argument, this time applied to a generic set $D$, concludes the proof: $\succ$ must be able to rank any arbitrary menu (i.e. it must be complete), if $\succ^*$ is allowed to violate monotonicity even at a single instance.

We wish to emphasize that this simple proof makes almost no use of existing representation theorems for menu preferences, such as the ones established in DLR. All we need are results regarding the existence of continuous and affine representations for $\succeq$ and $\succeq^*$. Beyond these auxiliary results, our argument essentially exploits the topology of the space to show how Continuity and Independence can work together to make the Cautious Deferral Axiom bite under any circumstance.

### 3.2 Necessity of the Axioms

The Independence Axiom is a key building block of most menu preference models. As previously mentioned, this axiom relies on the assumptions that the decision maker a) satisfies the standard Independence Axiom; b) is indifferent as to the timing of the resolution of uncertainty. This last assumption places strong restrictions on the way the decision maker evaluates a given menu. As shown by DLR, any complete preference satisfying Independence and Continuity can be seen “as if” it evaluates each menu in terms of some aggregation of the indirect utilities the set can provide for a certain collection of (Von Neumann-Morgenstern) utilities over lotteries.\footnote{In particular, there is a set $\mathcal{U}$ of Bernoulli utilities over prizes and an aggregator $V$ that is affine in $\oplus$ and that assigns to any set $A$ the value $V(\big\{\max_{a \in A} u \cdot a\big\})$.} Now consider an agent who is unable to compare two menus, say, $\{a\}$ and $\{a,b\}$. This must be because of a conflict between his multiple psychological rationales. For example, it might be that one of these, as in Sarver [53], induces him to prefer the singleton $\{a\}$, if he suffers from regret for not picking $b$ from $\{a,b\}$. At the same time, another one of these could suggest picking the larger menu in order to hedge against the possibility of a future change in preferences, as in the original model of Kreps [38]. In any case, we know that because of Independence, the disagreement must boil down to two opposite views on the indirect utility of $b$: some rationales will value it positively,
others will value it negatively.

At this point, one can see how the Cautious Deferral Axiom might be leveraged. For the completion to state that \( \{a, b\} \succ^* \{a\} \), it must necessarily give more weight to those rationales that value \( b \) positively than to the rest. If one can then show that this exercise can be repeated for every option, or at least for every relevant indirect utility, we are necessarily led to the result of overall monotonicity. The latter part is not obvious, which is what makes the result non-trivial.\(^{27}\) This rationalization suggests that if we wish to describe agents for which the cautious behavior suggested by Axiom 3 coexists with some indecisiveness and some preference for commitment, we must consider models in which either the relative value of an alternative can depend on the menu itself, or the value of a menu is not as strictly linked to the indirect utility of the available options.

In the Online Appendix (Section 2.3), we also show in a counterexample that the affinity implied by the Independence Axiom plays an essential role for Theorem 1, because it allows local properties to spread over the entire space. In this counterexample, we consider the natural relaxation of Independence to Indifference to Randomization, which requires that \( A \sim \text{co}(A) \) for all \( A \in X \). We show that this relaxation, which breaks the linearity behind the Independence Axiom while preserving the convexity, allows the Cautious Deferral Axiom to coexist with the non-monotonicity of \( \succ^* \). As such, our theorem is another illustration of the technical power of the Independence Axiom in a menu choice setting.

From the above discussion, one might be tempted to conclude that the result we obtain is essentially driven by the Independence Axiom. Yet, we show in two other counterexamples (Sections 2.1 and 2.2 of the Online Appendix) that our theorem is tight and requires the other ingredients of the model. In one counterexample, we show that closed continuity may not be relaxed or even replaced by the most common alternative for incomplete relations, open continuity. In another counterexample, we show that our requirement that the completion of \( \succ \) be proper is not innocuous, as it cannot be weakened to only requiring that \( A \succ B \) implies \( A \succ^* B \).

\(^{27}\)There are additional complexities that must be considered. For example one could value positively \( b \) when added to \( \{a\} \), but not when added to \( \{c\} \). This is possible under Independence whenever the collection of indirect utilities is infinite; see Krishna and Chatterjee [10].
3.3 Observability

The testable implications of our model, summarized by the Independence and Cautious Deferral axioms, crucially rely on the observability of the underlying psychological preference $\succ$. Yet, if an indecisive agent is forced to choose, and if in choosing he always follows a given completion $\succ^*$, one could reasonably object that there is no way we might recover $\succ$ from his choices. We wish to make two remarks about this matter.

First, the observation that incompleteness poses a challenge for revealed preference data is not new, and has been addressed by several papers in the literature (see Eliaz and Ok [20] and Danan and Ziegelmeyer [13]). This caveat is thus not specific to our work; rather, it is shared by a number of related models such as Gilboa et al. [31], Nehring [43], Kopylov [36] and Danan [11, 12]. Yet, as Gilboa et al. [31] also argue, our main goal is to make a conceptual point rather than to obtain a new representation. Our main point is here that if one agrees that a decision maker may have preferences that are innately incomplete, then one must be confronted with an unexpected tension between commitment and flexibility. By relying on this if exercise, our approach is reminiscent of behavioral economic theories, which in general stipulate that a certain behavioral aspect is connected to an unobservable psychological phenomenon and evaluate the consequences of this connection.

Secondly, although the key point of our contribution is conceptual, making testability a second-order concern, we believe that our main axiom can be falsified, as long as one is willing to consider a richer dataset. Although for many economic decisions we at best observe the constraint set and the chosen alternative, it is possible in more controlled settings such as the lab to give operational meaning to a DM’s indecisiveness. For example, a subject evaluating the choice between two menus $A$ and $B$ might be allowed to postpone his decision to a future date at a small cost, a strategy adopted by Danan and Ziegelmeyer [13] in a setting involving choices between lotteries; alternatively, a subject might be offered the option to pay a small price in exchange for a longer contemplation period. In both cases, one would interpret the request of a delay as an indication that the subject’s tastes over $(A, B)$ are incomplete.\[29\]

\[28\]This strategy was originally proposed in Danan [11] and [12]. Kopylov [35] proposes a similar elicitation device to identify the incomplete preference at the heart of his model.

\[29\]Another potential venue concerns the use of non-choice data. To wit, consider the following experimental design, implemented in Caplin et al. [7]. Subjects face a decision problem involving multiple options, which they must solve within a given time interval.
4 Related literature

There are a number of recent papers that investigate the completion and/or extension rules for incomplete preference relations in a variety of contexts. In particular, Gilboa et al. [31] and Kopylov [35] investigate such completions in the Anscombe-Aumann framework. The former builds a bridge between the classical models of Bewley [5] and Gilboa and Schmeidler [30] while the latter ties Bewley [5] to a model of ambiguity aversion known as the \( \epsilon \)-contamination model. Lehrer and Teper [40] consider a binary relation that is complete on a convex subset of the space of Anscombe-Aumann acts and propose an extension rule that generates Bewley [5] preferences as well as a completion rule that generates the MaxMin preferences. It is worth noting that most of these papers adopt a notion of completion which is weaker than the one we adopt in this paper, for they do not require the strict part of the incomplete relation to be preserved by its completion. This weaker notion of a completion is not suitable to our purposes because we interpret the underlying incomplete preference relation \( \succsim \) as representing the basic psychological tastes of the decision maker. In this respect, the revealed preferences of the agent, which represent a “completion” of \( \succsim \), must be perfectly in line with his psychological tastes.

Closer to the present work is that of Danan [12] who also works with a pair of preferences \( (\succ, \succ^*) \) defined over a generic set of menus (which may or may not consist of lotteries). The interpretation of this structure is the same as ours, that is, \( \succ \) stands for the psychological preferences of the agent while \( \succ^* \) refers to his behavioral (revealed) preferences. The main objective of that paper is however different from ours, namely, finding conditions under which \( \succ \) can be identified through the observation of \( \succ^* \). One of Danan’s main identifying conditions, which may appear as being quite similar to our Cautious Deferral axiom, can be expressed as: \( A \succ B \) if and only if \( A \cup B \succ^* A \) and \( A \cup B \succ^* B \). This condition is however much stronger than the Cautious Deferral Axiom for the following two reasons. First, preference for flexibility is required to be strict while our axiom also allows for indifference. For instance, consider a decision maker who cannot compare any two distinct menus (perhaps due to his

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The option a subject receives depends on the latest alternative he ticked on his screen at a randomly selected time in that interval. This setup allows Caplin and co-authors to collect adequately incentivized choice process data. One could envision that a similar design could be used to elicit incomplete preferences. For example, one could argue that the behavior of those subjects that keep switching between different options, even as the terminal time gets closer, is symptomatic of indecisiveness.
lack of information and/or interest) and hence is indifferent between any two menus. The preference structure of this agent is \((\succeq, X \times X)\), where \\
\[\succeq = \{(A, B) : A \neq B\}\]; it obviously satisfies the Cautious Deferral Axiom but fails Danan’s condition. Secondly, Danan identifies incomplete preferences with a strict desire for flexibility by requiring the implication to go in both directions. This identification strategy is used in Danan and Ziegelmeyer\[13\] in order to test the completeness axiom in a setting involving choices among menus of lotteries. In contrast, we do not impose the “if” part of Danan’s axiom which is fairly restrictive. For instance, consider an agent with complete DLR preferences such that \(A \cup B \succ^* A\) for at least two sets \(A\) and \(B\). While these preferences are standard in the literature on menu preferences, they do not satisfy Danan’s axiom. However, when \(\succeq\) is complete, the Cautious Deferral Axiom is trivially satisfied.

In a different setting, Arlegi and Nieto \[2\] also adopt a connecting condition, which presents similarities with the Cautious Deferral Axiom. Their condition, called Restricted Monotonicity, connects an asymmetric binary relation \(P\) defined on a finite set \(X\) of alternatives to a binary relation \(\succeq\) defined on the set of all menus from \(X\). This condition requires that for any two alternatives \(x\) and \(y\), \(xPy\) implies \(\{x, y\} \sim \{x\}\) and not \(xPy\) implies \(\{x, y\} \succ \{x\}\). If we identify \(xPy\) with \(\{x\} \succ \{y\}\), the second part of their axiom can be seen as a stronger version of Axiom 4b when restricted to singleton sets (for, as in Danan \[12\], preference for flexibility in the Restricted Monotonicity Axiom is required to be strict). Furthermore, we remain silent on what happens when two sets are comparable (the first part of that axiom); in particular, we do not impose that \(A \succeq B\) implies \(A \sim^* A \cup B\), the natural analog in our framework.

Finally, we note that several papers in the literature explore the connections between the internal conflict of a person and her tendency to defer choice. On the theoretical side, Gerasimou \[29\] considers a model in which choice deferral is driven by incomparability. This idea is captured by a choice correspondence which can be empty-valued. Closely related is the work of Buturak and Evren \[6\] who study the choice deferral phenomenon in a risky setting; they obtain a representation in which the decision-maker defers his choice from a set if and only if the option value of deferring (as an expected utility over subjective states) is larger than the current utility from the best available alternative in that set.\(^{30}\)

\(^{30}\)While the choice correspondence in Buturak and Evren \[6\] takes a specific value when deferral is chosen, the standard rationality axioms are only imposed in situations where the decision maker does not defer. In this respect, their approach is equivalent to the empty-set approach adopted by Gerasimou \[29\].
On the empirical side, Tversky and Shafir [55] show in one study that subjects are more likely to wait when the objects of choice are difficult to compare than when one of the alternatives appears to be clearly superior on all attributes. In a related study, they find that subjects’ propensity to wait in order to learn more about the various options available tends to increase if their choice is between two equally attractive options instead of just one option. In another study, Dhar [16] finds that choice deferral tends to increase when the differences in attributes among the available alternatives are small, as this makes alternatives harder to distinguish. Finally, Tykocinski and Ruffle [56] show that subjects may choose to postpone their choice even when they do not expect to receive additional information relevant to their choice, for instance if they need additional time for contemplation. Furthermore, they find a higher propensity to wait among subjects who express low confidence in committing themselves to a specific choice.

A Proof of Theorem 1

Part 1: Preliminaries Here we collect some results that will be useful for the proof of the main theorem. To begin with, we notice that, as established in DLR, Lemma 1, for any preference \(\succeq\) satisfying Axioms 1 and 2, \(A \sim co(A)\) for all \(A \in X\), where \(co(A)\) is the convex hull of \(A\).\(^\text{31}\) Since the space \(\hat{X}\) of closed and convex subsets of \(\Delta\) is a mixture space under \(\oplus\), it follows from Herstein and Milnor [34] that \(\succeq^*\) can be represented by a real valued function \(V\) that is continuous and affine in \(\oplus\), by which we mean that:

1) \(A \succeq^* B\) if and only if \(V(A) \geq V(B)\).
2) \(V(A^n) \to V(A)\) if \(A^n\) converges to \(A\) in the Hausdorff metric.
3) \(V(\alpha A \oplus (1 - \alpha) B) = \alpha V(A) + (1 - \alpha) V(B)\) for all \(\alpha\) in \((0, 1)\).

Secondly, Theorem 1 in Galaabaatar [27] ensures that there is an affine multi-utility representation \(W\) for \(\succeq\), where an affine multi-utility representation is a collection \(W\) of continuous and affine (in \(\oplus\)) real valued functions \(W\) on \(X\), such that

\[A \succeq B\] if and only if \(W(A) \geq W(B)\) for all \(W \in W\)

The following claim is the last preliminary result we need before we can provide a proof of Theorem 1:

Claim A.1 Let \(V : X \to \mathbb{R}\) be affine and continuous, and assume there are sets \(A_1, A_2, B_1, B_2 \in X\) such that \(A_i \subset B_i\) for \(i \in \{1, 2\}\), \(V(A_1) > V(B_1)\), and \(V(A_2) < V(B_2)\). Then we can find sets \(A, B, C \in X\) such that

\[A \subset B \subset C\] and \(V(A) > V(C) > V(B)\)

\(^{31}\)This property is often referred to as \textit{Indifference to Randomization}, because \(co(A)\) can be seen as the set of lotteries that can be obtained from \(A\) when we allow the agent to choose using a random strategy.
Proof: Let \( V(A_1) - V(B_1) = \epsilon \). By affinity of \( V \), we can find an \( \alpha \in (0,1) \) such that \( V(\alpha B_2 \oplus (1-\alpha)A_2) - V(A_2) < \epsilon \). Let \( B'_2 = \alpha B_2 \oplus (1-\alpha)A_2 \), and notice that \( A_2 \subset B'_2 \). Now let the three sets \( A, B, C \) be
\[
A = \frac{1}{2} A_1 \oplus \frac{1}{2} A_2, \quad B = \frac{1}{2} B_1 \oplus \frac{1}{2} A_2, \quad \text{and} \quad C = \frac{1}{2} B_1 \oplus \frac{1}{2} B'_2
\]
As required, \( A \subset B \subset C \). Moreover \( V(C) - V(B) = \frac{1}{2}(V(B'_2) - V(A_2)) > 0 \) and \( V(A) - V(C) = \frac{1}{2}\epsilon - \frac{1}{2}(V(B'_2) - V(A_2)) > 0 \) so that \( V(A) > V(C) > V(B) \). ■

Part 2: Proof of Theorem 1 To prove Theorem 1, it is sufficient to show that given Axioms 1-3, if \( \succcurlyeq \) is incomplete then \( \succcurlyeq^* \) must be monotone in set inclusion. Let \( V \) and \( W \) be the affine representations of \( \succcurlyeq^* \) and \( \succcurlyeq \) we just introduced, and assume, by contradiction, that \( A \succcurlyeq^* B \) for some \( A \subset B \). There are two cases we need to consider:

**Case 1:** \( A \succcurlyeq^* B \) for all \( A, B \) such that \( A \subset B \). In this case \( \succcurlyeq^* \) is always monotonically decreasing w.r.t. set inclusion, thus \( V(A) > V(A \cup B) \) for all \( A, B \in X \) such that \( V(A) > V(B) \). By assumption, there are two incomparable sets \( A_1, B_1 \). By Cautious Deferral then it must be that \( V(A_1) = V(B_1) \). But since \( \succcurlyeq^* \) is nonempty, we can find \( A, B \) such that \( V(A) > V(B) \). It then follows, from affinity of \( V \) and continuity of \( \succcurlyeq \) that we can find \( \alpha \in (0,1) \) big enough such that the sets \( A_0 = \alpha A_1 \oplus (1-\alpha)A_2 \) and \( B_0 = \alpha B_1 \oplus (1-\alpha)B_2 \) will satisfy \( V(A_0) > V(B_0) \) and \( A_0 \nsucccurlyeq B_0 \), leading to a contradiction.

**Case 2:** \( B \succcurlyeq^* A \) for some \( A \subset B \). If there are no \( A' \subset B' \) such that \( A' \succcurlyeq^* B' \) we are done. Otherwise, the conditions of Claim A.1 apply, thus we can find sets \( A \subset B \) such that \( A \succcurlyeq^* C \succcurlyeq^* B \). Since \( A \succcurlyeq^* B \) there are neighborhoods \( N'_A \) of \( A \) and \( N'_B \) of \( B \) such that \( A \succcurlyeq^* B \) for each pair \( (A, B) \in N'_A \times N'_B \). Since \( \cup \) is a continuous binary operation on \( X \), we can find neighborhoods \( N'_A \subseteq N'_A \) of \( A \) and \( N'_B \subseteq N'_B \) of \( B \) such that \( A \cup B \in N'_A \) for each pair \( (A, B) \in N'_A \times N'_B \). Notice that for each such pair \( \hat{A} \succcurlyeq^* A \cup B \).

Since \( \succcurlyeq^* \) is a proper completion of \( \succcurlyeq \), Cautious Deferral implies that \( A \succcurlyeq B \) for all \( (A, B) \in N'_A \times N'_B \). As a consequence \( W(A) > W(B) \) for every non constant \( W \) in the affine multi-utility representation of \( \succcurlyeq \), which allows us to set \( W(A) = 1 \) and \( W(B) = 0 \) for each non constant \( W \in \mathcal{W} \).

Since \( A \succcurlyeq^* C \succcurlyeq^* B \), there is an \( \alpha' \) such that \( \alpha' A \oplus (1-\alpha')B \sim^* C \) and \( \alpha A \oplus (1-\alpha)B \succcurlyeq^* C \) for all \( \alpha \in (\alpha',1) \). Since \( \alpha A \oplus (1-\alpha)B \subset C \) for all \( \alpha \in (0,1) \), Cautious Deferral implies \( \alpha A \oplus (1-\alpha)B \succcurlyeq C \) for all \( \alpha \in (\alpha',1) \). By continuity \( \alpha' A \oplus (1-\alpha')B \sim C \), which allows us to establish that \( W(C) = \alpha' \) for all non constant \( W \in \mathcal{W} \).

By continuity of \( \succcurlyeq^* \), there is an \( \alpha' \) such that \( A \succcurlyeq^* \alpha' C \oplus (1-\alpha') \Delta \succcurlyeq^* B \). Once again, we can find an \( \alpha'' \) such that \( \alpha'' A \oplus (1-\alpha'')B \sim^* \alpha' C \oplus (1-\alpha') \Delta \) and \( \alpha A \oplus (1-\alpha)B \succcurlyeq^* \alpha' C \oplus (1-\alpha') \Delta \) for all \( \alpha \in (\alpha'',1) \). Since \( \alpha A \oplus (1-\alpha)B \subseteq \alpha' C \oplus (1-\alpha') \Delta \) an argument similar to the previous leads us to conclude that \( \alpha'' A \oplus (1-\alpha'')B \sim \alpha' C \oplus (1-\alpha') \Delta \), so that \( W(\Delta) = \frac{\omega'' - \omega W(C)}{1-\omega''} \) for all non constant \( W \in \mathcal{W} \).

Finally, let \( D \) be a generic set in \( X \). We can find an \( \alpha' \) such that \( A \succcurlyeq^* \alpha' C \oplus (1-\alpha') \Delta \succcurlyeq^* \alpha' B \oplus (1-\alpha')D \). For some \( \alpha'' \) then \( \alpha'' A \oplus (1-\alpha'')\alpha' B \oplus (1-\alpha')D \sim^* \alpha' C \oplus (1-\alpha') \Delta \). Once again we can derive from this that \( \alpha'' A \oplus (1-\alpha'')\alpha' B \oplus (1-\alpha')D \sim \alpha' C \oplus (1-\alpha') \Delta \), which means all the \( W \)'s assign the same value to \( D \). Since this was done for a generic set \( D \), the implication is that \( \succcurlyeq \) is complete, a contradiction to our initial assumption.
References


