The inevitability of the "paradox of redistribution" in the allocation of voting weights

*Dietrich Fischer and Andrew Schotter**

Abstract

In voting bodies, when voting weights are reallocated, it may be observed that the voting power of some members, as measured by the Shapley-Shubik and Banzhaf power indices, increases while their voting weight decreases. By a simple constructive proof, this paper shows that such a "paradox of redistribution" can always occur in any voting body if the number of voters, n, is sufficiently large. Simulation results show that the paradox is quite frequent (up to 30 percent) and increases with n (at least for small n). Examples are given where the Banzhaf and Shapley-Shubik indices are not consistent in demonstrating the paradox.

Introduction

It is well known that the number or percentage of votes that a member in a voting body has is not a reliable index of his power. One example given by Brams and Affuso (1976) clearly illustrates this point. In a paper entitled, "Power and Size: A New Paradox," Brams and Affuso show that in the European Economic Community, when Ireland, Denmark, and Great Britain were admitted as members, the voting power (as measured by the Banzhaf, Coleman, and Shapley-Shubik power indices) of Luxembourg increased even though its fraction of the votes decreased.[1] They call this the "Paradox of New Members." More recently, Dreyer and Schotter (in progress) examined the distribution of voting power in the International Monetary Fund (as measured by the Banzhaf power index) following a redistribution of the voting weights. Again certain paradoxical results appeared which were counter to the intent of the planners and could have been avoided if proper

*Department of Economics, New York University. Professor Schotter's participation in this paper was made possible by partial support of the Office of Naval Research Contract N00014-76-C-0033 given to New York University.

attention was paid to the distinction between voting weights and voting power.

In their paper, Dreyer and Schotter show that in the reassignment of voting weights, thirty eight countries had their voting weights reduced yet gained in voting power when power was measured by the Banzhaf power index.[2] (See Appendix A for a summary of their results.) Their example differs from Brams and Affuso's in that at the International Monetary Fund no new members were added but rather the weights of the existing members were merely changed. Consequently, one could call their paradox the "Paradox of Redistribution," although Brams and Affuso's result could be considered a special case of theirs, in which one voter simply had a zero weight before the redistribution and a positive weight afterward, thereby becoming equivalent to a new member.

In this paper we consider the question of whether or not this paradox is possible for voting bodies of any size no matter what their distribution of voting weights or their decision rules when power is measured either by the Banzhaf or the Shapley-Shubik power index. In other words, is the paradox inevitable for any voting body, or are there vote distributions and decision rules for voting bodies which are "paradox proof," that is, for which we can find no other voting distribution which will increase some member's power while decreasing his voting weight? Our main results state that for Banzhaf power index, no paradox proof vote distribution exists for any voting body with $n \geq 6$ members, and for the Shapley-Shubik index, no paradox proof vote distribution exists for any voting body with $n \geq 7$. We will then show that when we restrict our voting rule to be a simple majority rule or when we restrict the voting distribution in a reasonable way by placing a lower bound on the weight of the smallest voter in the voting body, we are able to lower the size of the voting body in our results from $n \geq 6$ and $n \geq 7$ to $n \geq 4$. Simulation results are also reported which show that the paradox is not at all a rare occurence.

I. Power indices

Before we proceed to discuss our results, let us briefly review the power indices that we will be using. To do this, let N be the set of voters in a voting body indexed $i = 1, \ldots, n$, and let $w = (w_1, \ldots, w_n)$ be a vote distribution normalized such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$. The voting body is then fully described by an $n+1-$tuple $v = (d; w_1, \ldots w_n)$, where d is the decision rule of the body indicating the minimum fraction of votes that must be exceeded for the voting body to take collective action binding on all members,[3] and $(w_1, \ldots, w_n)$ is a vote distribution. Let S be any subset of voters $S \subseteq N$. Then we can define the value of coalition S as:

$$\nu(S) = 0 \text{ if } \sum_{i \in S} w_i \leq d,$$

$$\nu(S) = 1 \text{ if } \sum_{i \in S} w_i > d.$$

A voter is "critical" in a coalition S if his defection from that coalition changes the coalition from a winning to a losing coalition [i.e., $\nu(S) = 1$ and $\nu(S - \{i\}) = 0$].

The Banzhaf index for member i is then defined as:

$$p_B^i = \frac{\sum_S [\nu(S) - \nu(S - \{i\})]}{\sum_j \sum_S [\nu(S) - \nu(S - \{j\})]}$$

This index, then, describes the proportion of critical defections of member i. It follows that $\sum_{i=1}^{n} p_B^i = 1$.[4]

The Shapley-Shubik index for a member i is slightly more complex. It concerns itself with the proportion of permutations of the n members in which i's defection from a winning coalition is critical. More formally, it can be written as:

$$p_{SS}^i = \sum_S \left[\frac{(s-1)!\,(n-s)!}{n!}\right] [\nu(S) - \nu(S - \{i\})]$$

where s is the number of members in the subset S and n is the total number of members in the voting body.

II. The paradox of redistribution for a three member voting body

Consider the following voting body:

$$\nu = \left(\frac{70}{100}; \frac{55}{100}, \frac{35}{100}, \frac{10}{100}\right),$$

where 70/100 is the decision rule and (55/100, 35/100, 10/100) is the vote distribution. The Banzhaf and Shapley-Shubik power indices associated with this voting body are both (1/2, 1/2, 0). Now let us redistribute the votes, keeping the decision rule the same, so that the following voting body is determined:

$$\nu' = \left(\frac{70}{100}; \frac{50}{100}, \frac{25}{100}, \frac{25}{100}\right).$$

Here the Banzhaf index[5] is (3/5, 1/5, 1/5), while the Shapley-Shubik index is (2/3, 1/6, 1/6), showing that although member 1's voting weight decreased from 55/100 to 50/100, his power increased.[6]

Now it might be interesting to ask whether in a three member voting body there exist vote distributions which are paradox proof in the sense that any redistribution of the votes which starts at one of these distributions must give a voter less voting power if it gives him less weight. The answer is yes, and as a matter of fact there are an infinite number of such vote distributions, as is illustrated in Appendix B. For the purpose of illustration, however, consider the following voting body, $v = (70/100; 55/100, 23/100, 22/100)$. Here the power distribution is (3/5, 1/5, 1/5) if the Banzhaf index is used, and (2/3, 1/6, 1/6) if the Shapley-Shubik index is used. The reader can check for himself that it is not possible to give a voter more power by diminishing his voting weight. Consequently, the vote distribution (55/100, 23/100, 22/100) is a paradox proof distribution for a three-member voting body with decision rule $d = 70/100$.

III. The inevitability of the paradox of redistribution

The question we investigate in this paper, then, is the circumstances under which the paradox illustrated above is inevitable. To do this we must demonstrate how, given a vote distribution $w = (w_1, \ldots, w_n)$ and a voting rule d, we can construct a new vote distribution $w' = (w'_1, \ldots, w'_n)$ which gives at least one voter a smaller proportion of the vote yet gives him more power, when power is measured by either the Banzhaf or the Shapley-Shubik power index. The following propositions do just that.

Proposition 1: For voting bodies with $n \geq 6$, a paradox is always possible no matter what initial vote distribution exists, when power is defined by the Banzhaf index.

Proof of Proposition 1

1. For any voting rule d and any weight w_1, of voter 1, his maximum power is at least 1/5, no matter how small w_1 is. He can achieve this power by assigning weight $w_2 = d - w_1/2 - \delta/2$ to voter 2, $w_3 = 1 - d - w_1/2 - \delta/2$ to voter 3, and $w_4 = \ldots = w_n = \delta/(n-3)$ to the remaining voters where δ is arbitrarily small.[7]. In particular $w_4 + \ldots + w_n = \delta < w_1/2 + \delta/2$, that is, $\delta < w_1$, so that the voters 4, ..., n are all dummies with zero voting power.

2. If there are 6 voters, the weakest of them has at most power 1/6 (since the power indices add up to 1). By point 1 of the proof, the weakest voter can now redistribute the weights of the other voters and increase his power to 1/5, keeping his voting weight constant. However, since he can achieve a power of 1/5 with an arbitrarily *small* voting weight, he can in fact *decrease* his weight to *any* small positive value and still increase his power. Q.E.D.

Proposition 2: For voting bodies with $n \geq 7$, a paradox is always possible no matter what initial vote distribution exists, when power is defined by the Shapley-Shubik index.

Proof of Proposition 2

Analogous to the proof of Propostion 1.

The import of these theorems is that for any voting body $v = (d; w_1, \ldots, w_n)$ – where $n \geq 6$ for the Banzhaf case and $n \geq 7$ for the Shapley-Shubik case – any initial vote distribution is subject to the paradox of redistribution in that there always exists another vote distribution such that in the redistribution of voting weights, at least one player's weight is decreased while his voting power is increased. It is in this sense that the paradox is inevitable.

One shortcoming of our results is that they do not give us any indication of the minimum size of the redistribution necessary to create the paradox. Clearly, our results would be more disturbing if for any given initial distribution of votes there was another distribution within a small neighborhood of the original which would create the paradox. It is interesting to note that in the International Monetary Fund example summarized in Appendix A, the redistribution of voting weights was not a "drastic" one. Also, our simulation results given in Section VI demonstrate that the occurrence of the paradox is not a rare event.

IV. Some further results

Our results so far are rather general in that we do not constrain the decision rule used in our voting bodies in any way. However, the most common voting rule is the simple majority voting rule and when this rule is used, it is possible to show that both for the Banzhaf and the Shapley-Shubik power indices no paradox proof vote distributions exist for $n \geq 4$.

Proposition 3: If the voting rule used is the simple majority voting rule (i.e., if $d = 1/2$), then for $n \geq 4$ a paradox is always possible.

Proof of Proposition 3

Let 1 be the weakest voter. (Every voter's weight is always positive, thus $w_1 > 0$.) Voter 1 can now achieve a power of 1/3 by distributing the weights as w_1, $w_2 = (1 - w_1 - \delta)/2$, $w_3 = w_2$, $w_4 = \delta/(n-3) = \ldots = w_n$ with $\delta > w_1$. Then voters 4, ..., n all have zero voting power and voters 1, 2 and 3 have power 1/3 each. This is true both for the Banzhaf and the Shapley-Shubik power indices.

With four or more voters in a voting body, the weakest voter has, at most, power 1/4 (in general $1/n \leq 1/4$). But the weakest voter can increase his power to 1/3 by redistributing the voting weights of the remaining players as indicated while keeping his voting weight the same. However, since voter 1 can achieve a power of 1/3 even with an arbitrarily small voting weight, he can in fact decrease his voting weight and still increase his power. Thus a paradox of redistribution always exists for $n \geq 4$ and $d = 1/2$. Q.E.D.

Instead of restricting the voting rule as we did in Proposition 3, we could have placed a restriction on the vote distribution. One restriction would be to constrain the weight of the smallest voter. If we make the restriction that the voter with the smallest voting weight (say voter i) has a weight $w_i > 2d - 1$ initially (which in the case of $d = 51\%$ merely requires that he have more than 2% of the vote), then it is possible to prove that no paradox proof vote distributions exist for $n \geq 4$ for both the Banzhaf and Shapley-Shubik power indices.

Proposition 4: If the decision rule is d and if the voter with the smallest voting weight in the voting body $v = (d; w_1, \ldots, w_n)$ has a weight greater than $2d - 1$, then with $n \geq 4$ (and either the Banzhaf or the Shapley-Shubik index) a paradox is always possible.

Proof of Proposition 4

The weakest voter (say voter 1) has at most power $1/n \leq 1/4$. We can increase his power to 1/3 by assigning weights $w_1, w_2 = w_3 = \dfrac{1 - w_1 - \delta}{2}$, $w_4 = \ldots = w_n = \delta/(n-3)$ with $w_1 + w_2 = (1 + w_1 - \delta)/2 > d$ or $\delta < w_1 - (2d-1)$, where it is always possible to choose $\delta > 0$, by assumption. Q.E.D.

V. The reverse paradox

Instead of looking for a paradox of the kind discussed up to now, where a voter whose voting weight is decreased gains more power, we may also ask whether the reverse situation exists. Are there cases, where a voter gains in terms of voting weight, but loses in voting power? The answer is, of course, yes. For whenever there is a paradox of the first kind in going from a voting weight distribution $w = (w_1, \ldots, w_n)$ to a new distribution $w' = (w'_1, \ldots, w'_n)$, i.e., if some voter i is assigned a lower weight $w'_i < w_i$ but gains in power $(p'_i > p_i)$,[8] then a reverse paradox is observed in going from the new voting weight distribution w' to the old distribution w, since then voter i is assigned a greater voting weight but loses in terms of power.

The question arises whether this reverse kind of paradox is also inevitable; i.e., given any initial voting weight distribution w, does there always exist a new distribution w' which exhibits a reverse paradox? The answer is clearly no if we allow dictators to be present. For example, assume one voter has a weight exceeding the required majority (say $w_1 > d$), that is, where one voter is a dictator with power equal to 1, and all the remaining voters are dummies with zero voting power. Then it is clear that the dictator (voter 1) is the only one who can possibly lose power, but he certainly will not do so if his voting weight is increased. All the other voters can only increase their power or keep it unchanged, regardless of what happens to their voting weights. Thus, we see that in this case no reverse paradox is possible.

Fortunately, however, we do not have to limit ourselves to such extreme cases, for even when we exclude the possibility that one of the voters is a dictator, there still exist "paradox proof" voting weight distributions in the reverse sense, for any number of voters n, provided that the required majority d is sufficiently high. We shall see that a voting body in which each member has veto power is paradox proof in the reverse sense. To show this, we make use of the following:

Lemma 1: Any voter who has *veto power* (i.e., any voter i whose weight is $w_i \geq 1 - d$) has at least as much power as any other voter in the voting body, when power is measured by either the Banzhaf or the Shapley-Shubik power index.

Proof of Lemma 1

Assume voter i has veto power. Then he must be a member of *every* winning coalition S since $\sum_{j \in S} w_j > d$ can hold only if $i \in S$. Also, he is critical in every coalition, since $\sum_{j \in S'} w_j \leq d$ for every S' which does not include voter i.

The Banzhaf power index for voter i:

$$p_B^i = \sum_S [v(S) - v(S - \{i\})] / \sum_j \sum_S [v(S) - v(S - \{j\})],$$

is directly proportional to the number of winning coalitions S in which i is a critical member. No other voter can possibly belong to more winning coalitions or be a critical member in more winning coalitions, since i belongs to every winning coalition and is critical in each one of them. Therefore, no other voter can have a higher Banzhaf power index than i.

The Shapley-Shubik power index of voter i:

$$p_{SS}^i = \frac{1}{n!} \sum_S [(s-1)!\,(n-s)!]\ [v(S) - v(S - \{i\})],$$

is proportional to the number of winning coalitions S in which i is critical, multiplied by a certain factor $(s-1)!\,(n-s)!$, whose size depends only on the size s of the respective winning coalition S, but not on the voter i. That is, given some winning coalition S, every critical member of S derives the same measure of power from membership in S, when power is measured by the Shapley-Shubik index. Thus, the same argument as before again applies, namely that voter i is at least as powerful as any other voter, since he is a critical member of any winning coalition. No other voter can outdo him in this respect.

This completes the proof of Lemma 1.

Corollary: A voter who has veto power in a voting body with n members has at least power $1/n$, when power is measured by either the Banzhaf or the Shapley-Shubik index.

Proof of the Corollary to Lemma 1

Suppose to the contrary that voter i has veto power, and that $p_B^i < 1/n$ (or $p_{SS}^i < 1/n$). Then there must exist at least one other voter whose power index is larger than $1/n$, for otherwise the power indices of all voters would sum up to less than 1. But according to Lemma 2, it is impossible to find another voter whose power index is greater than that of i. This completes the proof of the corollary.

Proposition 5: Any voting weight distribution in which every member has veto power is paradox proof in the reverse sense.

Proof of Proposition 5

According to the corollary of Lemma 1, every voter in a voting body where each voter has veto power must have a power of at least $1/n$ (measured by either the Banzhaf or the Shapley-Shubik index), and since the power indices sum up to 1, the power of each voter is exactly $1/n$. If now the voting weight of any voter is increased, he still keeps his veto power, regardless of how voting weights are redistributed among the remaining voters. That is, his power is at least $1/n$, and can therefore not decrease. Thus, it is impossible to find any voter whose power decreases when his voting weight increases. This completes the proof of Proposition 5.

In light of Proposition 5, it may be interesting to note under what conditions a voting weight distribution can exist in which all voters have veto power. We have:

Proposition 6: A voting weight distribution $w = (w_1, \ldots, w_n)$ in a voting body $v = (d; w_1, \ldots, w_n)$ which gives every voter a veto can exist if and only if $d \geq 1 - 1/n$.

Proof of Proposition 6

a. Necessity. The weakest member in a voting body of size n has a voting weight of at most $1/n$ (otherwise the weights would sum up to more than 1). If he is to have veto power, the decision rule must be $d \geq 1 - 1/n$.

b. Sufficiency. If $d \geq 1 - 1/n$, then there exists at least one weight distribution which gives each voter a veto, namely $w = (1/n, \ldots, 1/n)$. (In general, there exists a continuum of such distributions, the more, the larger d is.)

Does the result that a reverse paradox is not inevitable mean that it is less likely to occur when voting weights are reallocated than a paradox of the first kind? Not at all. In fact, from the symmetric relationship between a paradox and a reverse paradox, which was discussed at the beginning of this section, it follows that for any number n of voters and any decision rule d a reverse paradox is exactly as likely to occur as a paradox of the kind discussed earlier, provided that the voting weight distributions w and w' before and after redistribution are independent random selections from the same probability distribution. In the next section, we report some simulation results which allow us to estimate the frequency with which one may find a paradox of redistribution.

VI. The frequency of paradoxes of redistribution: a Monte Carlo simulation

In proving Propositions 1 and 2, where we showed that if the number of voters is $n \geq 6$ for the Banzhaf index and $n \geq 7$ for the Shapley-Shubik index, a paradox of redistribution is always possible, we used rather special redistribution patterns, giving very small voting weights to all but a few members of the voting body. The question naturally arises whether such paradoxical situations occur with any significant frequency, or whether we are merely attacking windmills. Another question one may ask is whether the frequency of a paradox will tend to increase or decrease as the number n of voters increases.

To answer these questions, we have simulated a large number of random redistributions of voting weights, for various numbers of voters n and decision rules d, and counted the number of paradoxes observed. For each redistribution, we selected two random vectors of voting weights, $w = (w_1, \ldots, w_n)$, and $w' = (w'_1, \ldots, w'_n)$, which were independent and uniformly distributed on the $(n-1)$-dimensional unit simplex given by $w_1 + \ldots + w_n = 1$, $w_i \geq 0$ $(i = 1, \ldots, n)$.[9] For each such pair of voting weight distributions we computed the Banzhaf and the Shapley-Shubik power index before and after redistribution and tested whether any voter had increased power with a smaller voting weight, or the reverse.[10] Table 1 summarizes the results for the Banzhaf power index and Table 2 for the Shapley-Shubik power index.[11]

Table 1. Simulation of paradoxes for the Banzhaf power index: number of simulation runs performed, observed number of paradoxes of redistribution,[1] and 95% confidence interval on the occurrence of a paradox, for various numbers *n* of voters and decision rules *d*.

Number *n* of voters	Number of trials[2]	*d* (Decision rule)				
		0.5	0.6	0.7	0.9	0.9
3	10,000	617 .057–.066	638 .059–.069	453 .041–.049	688 .064–.074	1384 .132–.145
4	5,000	689 .128–.147	663 .123–.142	600 .111–.129	740 .138–.158	981 .185–.207
5	2,000	503 .232–.271	364 .165–.199	404 .184–.220	420 .192–.288	520 .241–.279
6	1,000	295 .267–.323	238 .212–.264	225 .199–.251	274 .246–.302	285 .257–.313
7	500	153 .266–.346	110 .184–.256	118 .199–.273	129 .220–.296	184 .326–.410
8	250	77 .251–.365	47 .140–.236	53 .161–.263	76 .247–.361	85 .281–.399

Notes.

1. Very similar figures were obtained for the occurrence of the reverse paradox. But these could not be used to improve the estimate of the probability of occurrence of a paradox, since they were based on the same pair of voting weight distributions and did not constitute an independent observation.
2. For larger numbers *n* of voters, fewer simulation runs were performed to save computer time.

Table 2. Simulation of paradoxes for the Shapley-Shubik power index: observed number and 95% confidence interval on the probability of occurrence.

Number n of voters	Number of trials	d (Decision rule)				
		0.5	0.6	0.7	0.8	0.9
3	10,000	617 .057–.066	638 .059–.069	453 .041–.049	688 .064–.074	1384 .132–.145
4	5,000	689 .128–.147	630 .117–.135	548 .101–.118	706 .132–.151	966 .182–.204
5	2,000	503 .232–.271	371 .168–.203	388 .177–.211	426 .195–.231	519 .240–.279
6	1,000	264 .237–.291	228 .202–.254	202 .177–.227	244 .217–.271	301 .273–.329
7	500	141 .243–.321	111 .186–.258	122 .206–.282	140 .241–.319	182 .322–.406
8	250	67 .213–.323	56 .172–.276	54 .165–.267	69 .221–.331	90 .300–.420

Figure 1 graphically represents a portion of the results listed in Table 1. These results clearly show that the frequency of a paradox is by no means negligible, reaching as high as 30 percent for $n = 6$ voters, under the radical redistribution of voting weights assumed here. We can also conclude that the frequency of a paradox increases with the number n of voters, at least for the range of relatively small n explored here. With respect to the parameter d (the decision rule) the probability of the occurence of a paradox apparently drops at first, reaches a minimum for about $d = 0.7$, and then increases again as d approaches 1. We do not have any simple and plausible explanation for this behavior.

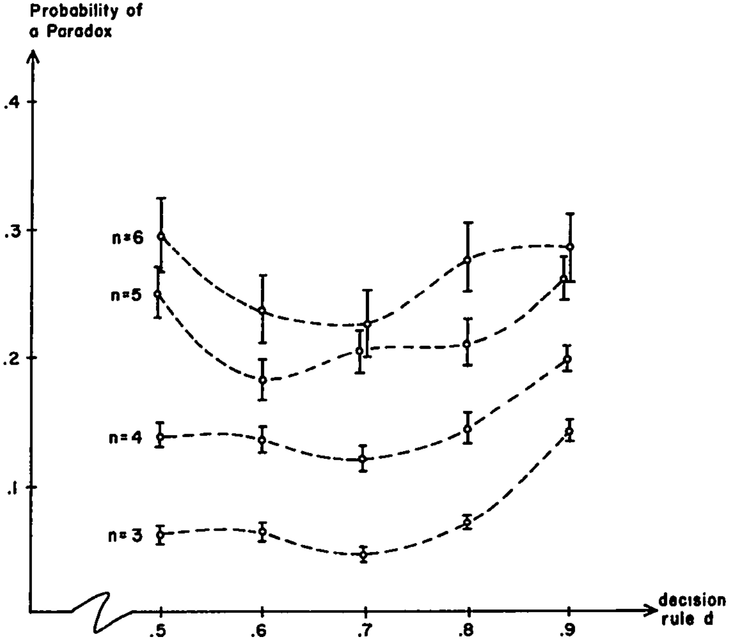

Figure 1. 95% confidence intervals for the probability of a paradox of redistribution for the Banzhaf power index, for various numbers n of voters and decision rules d.

In order to verify at least one entry in Tables 1 and 2, we have computed the theoretical probability of a paradox for $n = 3$ and $d = 0.5$. As is shown in Appendix C, this probability is equal to $1/16 = 0.0625$. We would therefore expect that approximately 625 paradoxes would occur among 10,000 randomly selected voting weight redistributions. The observed number of

617 paradoxes lies well within the expected range.

Finally, it may be interesting to note that there are cases where a paradox occurs when power is measured by the Banzhaf index, but not as measured by the Shapley-Shubik index, and vice versa. Table 3 gives an example each where such a discrepancy occurs.[12]

Table 3. Two examples of voting weight distributions with $n = 4$ voters and decision rule $d = 0.6$[1]

		Case (a)				Case (b)			
		$i = 1$	2	3	4	1	2	3	4
Voting weights	w_i	.378	.202	.254	.166	.066	.081	.410	.444
	w_i'	.279	.187	.071	.464	.358	.089	.401	.152
Banzhaf power indices	p_B^i	1/3	1/6	1/3	1/6	0	0	1/2	1/2
	$p_B^{i\prime}$	1/5	1/5*	0	3/5	3/10	1/10	1/2	1/10
Shapley-Shubik power indices	p_{SS}^i	1/3	1/6	1/3	1/6	0	0	1/2	1/2
	$p_{SS}^{i\prime}$	1/6	1/6	0	2/3	1/4	1/12	7/12*	1/12

Note.

1. In Case (a) a paradox is observed when power is measured by the Banzhaf index, but not when measured by the Shapley-Shubik index. Case (b) shows the opposite. The power indices which cause the paradox are marked by an asterisk.

VII. Conclusions

In this paper we have seen that the "Paradox of Redistribution" reported by Dreyer and Schotter (in progress) is an inevitable paradox of power, in the sense that for any voting body (with $n \geq 6$ in the Banzhaf case and $n \geq 7$ in the Shapley-Shubik case) with an initial vote distribution $w = (w_1, \ldots, w_n)$, there exists another vote distribution $w' = (w_1', \ldots, w_n')$ which gives at least one voter a smaller proportion of the vote yet gives him more power, when power is measured either by the Banzhaf or the Shapley-Shubik power indices. Furthermore, we have found that the paradox does in fact occur with surprising frequency (about 30 percent in a body of $n = 6$ voters with simple majority rule, if the voting weights before and after redistribution are assumed to be purely random).

These results should then serve as a warning to organizations that are planning vote redistributions in their voting bodies, since the actual result of the redistribution may be opposite to the intended goal of the planners. Such unintended results can only be avoided if the distinction between voting power and voting weights is recognized.

Appendix A

Occurrence of the paradox of redistribution in the international monetary fund[1]

Country	Present voting system $d = .80$		Proposed voting system $d = .85$	
	% of Vote	% of Power	% of Vote	% of Power
Luxembourg	.14	.0020	.13	.0021
Papua New Guinea	.14	.0020	.13	.0021
Jordan	.15	.0021	.13	.0021
Honduras	.15	.0022	.14	.0023
Cyprus	.16	.0022	.14	.0023
Malagasy Republic	.16	.0022	.14	.0023
Ethiopia	.16	.0023	.14	.0023
Liberia	.17	.0024	.14	.0024
Yemen (P.D.R.)	.17	.0024	.16	.0025
Costa Rica	.18	.0025	.16	.0025
Cameroon	.19	.0026	.17	.0027
Guatemala	.19	.0027	.18	.0029
Panama	.19	.00271	.17	.00274
Bahamas	.14	.0200	.13	.0022
Dominican Republic	.21	.0030	.19	.0031
Kenya	.23	.0032	.22	.0036
Tunisia	.23	.0032	.21	.0034
Syria	.23	.0033	.21	.0034
Jamaica	.24	.0034	.23	.0038
Burma	.26	.0037	.23	.0038
Trinidad & Tobago	.27	.0039	.25	.0041
Uruguay	.29	.0041	.26	.0042
Sudan	.30	.0043	.27	.0044
Ghana	.35	.0049	.31	.0051
Sri Lanka	.38	.0054	.34	.0056
Iraq	.41	.0059	.39	.0064
Morocco	.43	.0061	.41	.0067
Zaire	.43	.0061	.42	.0068
Ireland	.45	.0064	.43	.0069
Peru	.46	.0065	.45	.0073
Bangladesh	.46	.0066	.42	.0068
Turkey	.54	.0078	.53	.0086
Egypt	.66	.0074	.60	.0096
Romania	.66	.0045	.64	.0120
Pakistan	.80	.0114	.73	.0116
Norway	.82	.0116	.76	.0120
Denmark	.88	.0125	.79	.01255
Austria	.91	.0129	.84	.0132

Note 1.

These results compare the Banzahf indices for a redistribution of voting weights with $d = .80$ in the existing system and $d = .85$ in the proposed system as the rules prescribe. As Brams and Affuso (1976) point out in their article and as our examples in Sections II and VI illustrate, however, this result is not an artifact of the change in the decision rule but could occur with the same decision rule used in the existing and proposed systems.

Appendix B

Paradox proof vote distributions for three voters

In a voting body with $n = 3$ voters, there always exists a set of distributions which are paradox proof, no matter what the voting rule d is. In the following three figures the paradox proof vote distributions are represented by shaded areas, in barycentric coordinates on the open simplex with $w_1 + w_2 + w_3 = 1$ and $w_i > 0$ $(i = 1, 2, 3)$. Three cases must be distinguished for $1/2 \leq d < 3/5$ (Figure 2a), $3/5 \leq d < 2/3$ (Figure 2b), and $2/3 \leq d < 1$ (Figure 2c). For $1/2 \leq d < 3/5$, there is a single contiguous region of paradox proof points in the center. For $3/5 \leq d < 2/3$, there are 4 such regions. For $2/3 \leq d < 1$, there are 3 such regions.

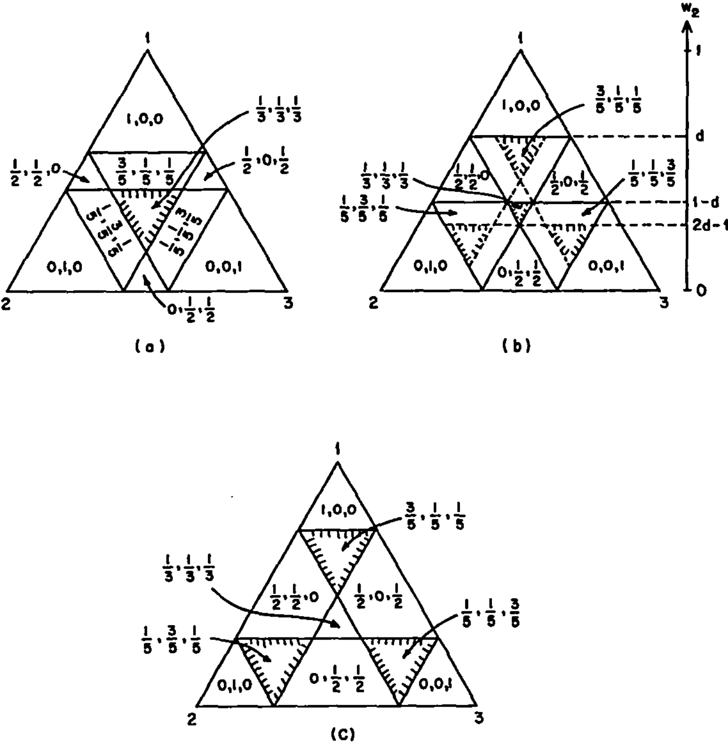

Figure 2. Paradox proof voting weight distributions (shaded areas) for $n = 3$ voters on the unit simplex $Z = \{w \mid \Sigma w_i = 1, w_i > 0, i = 1, 2, 3\}$ for a) $1/2 \leq d < 3/5$; b) $3/5 \leq d < 2/3$; c) $2/3 \leq d < 1$.

The numbers in the various areas give the Banzhaf power index for voters 1, 2 and 3, respectively. For the Shapley-Shubik index, the power indices and the dividing lines between the various areas remain the same, except that the power distributions such as $p_B = (3/5, 1/5, 1/5)$ etc., are replaced by $p_{SS} = (2/3, 1/6, 1/6)$. It is easy to verify that for the indicated points in the shaded areas no voter can increase his power if his weight remains constant (for example on a horizontal line for voter 1).

Appendix C

Calculation of the theoretical probability for the occurrence of a paradox

To verify the first entry in Tables 1 and 2, we derived the theoretical probability for the occurence of a paradox of redistribution when $n = 3$ and $d = 0.5$. In this case, the vector of power indices (as measured by either the Banzhaf or the Shapley-Shubik index) can be any of the 4 possible combinations shown in Figure 3(a). For a paradox to occur, one of the voters must increase his power index from 0 to 1/3 while his voting weight is reduced. If the initial distribution of voting weights is given by the vector $w^* = (w_1^*, w_2^*, w_3^*)$ shown in Figure 3(b), then a paradox occurs if w' lies in the shaded area. (For example, if w' lies in the small shaded triangle at the bottom of Figure 3(b), then the weight of voter 1 was reduced, while his power increased from 0 to 1/3.)

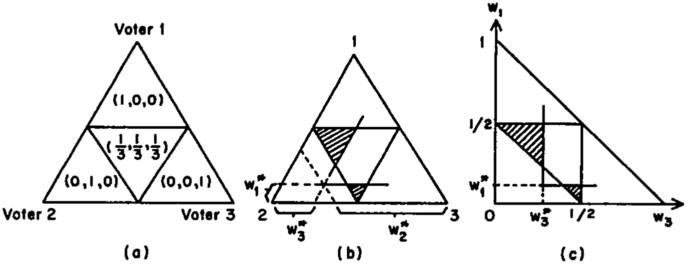

Figure 3. Starting from the voting weight distribution $w^* = (w_1^*, w_2^*, w_3^*)$, a paradox occurs if w' lies in the shaded area.

Given that the initial weight distribution is w^*, the probability of a paradox is equal to the ratio between the sum of the two shaded areas and the area of the unit simplex*:

$$p(w^*) = (w_1^*)^2 + (w_3^*)^2.$$

We can simplify the problem by eliminating the coordinate $w_2 = 1 - w_1 - w_3$ and transform the unit simplex into the triangle shown in Figure 3(c), which is defined by $w_1 \geq 0$, $w_3 \geq 0$ and $w_1 + w_3 \leq 1$. The probability that the initial weight distribution

*We recall here that we assumed that each point on the unit simplex has equal probability of being selected for w and w', and that w and w' are stochastically independent.

lies in the lower left triangle defined by $w_1 + w_3 \leq 1/2$ *and* that a paradox occurs is then given by the expression:

$$\tilde{p} = \int_{w_1=0}^{1/2} \int_{w_3=0}^{1/2-w_1} p(w)dw_3\,dw_1 \Big/ \int_{w_1=0}^{1} \int_{w_3=0}^{1-w_1} dw_3\,dw_1$$

where $p(w) = w_1^2 + w_3^2$. After some simplification, this yields $\tilde{P} = (1/96) / (1/2) = 1/48$. A symmetry consideration shows that a paradox can also occur, with this same probability $\tilde{P}$, when w lies in any of the other two corner areas. Thus, we find $P = 3\tilde{P} = 1/16$ for the probability of a paradox.

A similar expression can be derived for the probability of a reverse paradox, and also yields 1/16. But this follows directly from the symmetry between a paradox and a reverse paradox.

Notes

1.

Joseph Raanan (1976) has proven that his paradox is not a peculiarity of the particular power indices used by Brams and Affuso, but it is true for any power index satisfying four reasonable axioms. Consequently, what Brams and Affuso have illustrated is a true paradox of power and not a paradox of power indices.

2.

In the process of redistributing the voting power, the IMF also changed the voting rule from a necessary 80 percent to a necessary 85 percent majority. As our paper illustrates, however, the paradox is not an artifact of this change and could occur without it.

3.

d must be at least equal to 1/2, otherwise two or more conflicting decisions could be adopted. The requirement $\sum_{i \in S} w_i > d$ for a winning coalition is slightly different from standard usage which only requires $\Sigma w_i \geq d$. The latter rule is usually applied to distributions of integer numbers of votes. In the case of continuous voting weight distributions, with which we are concerned here, $\Sigma w_i > d$ seems a more natural condition for a winning coalition. We shall frequently refer to the "simple majority rule" which is represented by $d = 1/2$ using the above definition. With the standard definition there is no value of d which precisely describes the simple majority rule, once continuous voting weights are admitted. Corporations often use a 51 percent rule for a winning coalition, but his is more than a simple majority.

4.

For a discussion of the relationship of the Banzhaf index to the Coleman index, see Brams and Affuso (1976, pp. 31–34).

5.

Voter 1 is critical in the coalition $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 3\}$, Voter 2 is critical in coalition $\{1, 2\}$and voter 3 is critical in coalition $\{1, 3\}$. This directly yields the Banzhaf index given.

6.

The paradox of new members can be demonstrated as follows. Consider the following voting body, ν = (70/100; 60/100, 40/100, 0). This is in essence a two voter voting body. Now redistribute the votes so that the following voting body is determined: ν' = (70,100; 50/100, 25/100, 25/100). The original Banzhaf power index is (1/2, 1/2, 0), as is the original Shapley-Shubik index, while the new Banzhaf index is (3/5, 1/5, 1/5) and the new Shapley-Shubik index is (2/3, 1/6, 1/6) which illustrates the "Paradox of New Members".

7.

Suppose, for example, w_1 = 1%, d = 70% and there are n = 8 voters. Then the first voter has a Banzhaf power index of 1/5 if the voting weights are distributed as (using δ = 0.5%) w_1 = 1%, w_2 = 69.25%, w_3 = 29.25%, $w_4 = w_5 = \ldots = w_8$ = 0.1%.

8.

p_i and p_i' stand for either the Banzhaf or the Shapley-Shubik power index.

9.

A method to generate a random vector w which is uniformly distributed on the $(n-1)$-dimensional unit simplex is the following. We first choose $n-1$ random numbers $y_1, \ldots, y_{n-1}$ which are independent and uniformly distributed in the interval $0 \leq y_i \leq 1$ $(i = 1, \ldots, n-1)$. Then we permute the y_i to bring them into non-decreasing order $0 \leq y_1' < y_2' \leq \ldots \leq y_{n-1}' \leq 1$. A uniformly distributed random vector on the unit simplex is then given by $w_1 = y_1'$, $w_2 = y_2' - y_1', \ldots, w_{n-1} = y_{n-1}' - y_{n-2}'$, $w_n = 1 - y_{n-1}'$. To see this, we note that the vector $y = (y_1, \ldots, y_{n-1})$ is uniformly distributed on the $(n-1)$-dimensional hypercube $C_{n-1} = \{y \mid 0 \leq y \leq 1, i = 1, \ldots, n\}$. Rearranging the coordinates into increasing order maps this point $y \in C_{n-1}$ into a point $y' \in Z_{n-1}$ on the $(n-1)$-dimensional simplex, $Z_{n-1}' = \{y' \mid 0 \leq y_2' \leq \ldots \leq y_{n-1}' \leq 1\}$, where $Z_{n-1}' \subset C_{n-1}$. (Z_{n-1}' is not a *unit* simplex.) y' is uniformly distributed on Z_{n-1}', that is, every point in Z_{n-1}' has equal probability of being selected. The transformation $w_1 = y_1'$, $w_2 = y_2' - y_1', \ldots, w_n = 1 - y_{n-1}'$ mapsZ_{n-1}' into the $n-1$ dimensional unit simplex $Z_{n-1} = \{w \mid \sum_i w_i = 1, w_i \geq 0, i = 1, \ldots, n\}$, keeping the property that each point in Z_{n-1} has equal probability of being selected.

10.

Efficient algorithms to compute the Banzhaf and the Shapley-Shubik power index using generating functions are given in Brams and Affuso (1976).

11.

A 95% confidence interval of the probability of the occurrence of a paradox was calculated by using the fact that the number n_p of paradoxes has a binominal distribution with an unknown probability p and a number m of trials equal to the number of simulation runs performed. An unbiased estimator for p is $\hat{p} = n_p/m$, and the standard error of the estimation is $\epsilon = \sqrt{\hat{p}(1-\hat{p})/m}$. A 95% confidence interval was taken as $(\hat{p} - 1.96\epsilon, \hat{p} + 1.96\epsilon)$, using the normal approximation to the binomial distribution.

12.

In Appendix B, we will see that for n = 3 voters a paradox for the Banzhaf index occurs if and only if it also occurs for the Shapley-Shubik index. But for $n \geq 4$ this need no longer be true, and therefore the entries in Tables 1 and 2 differ in some places for $n \geq 4$.

References

Brams, S. J., and Affuso, P. J. "Power and Size: A New Paradox," *Theory and Decision*, 7, 1976, pp. 29–56.

Dreyer, J., and Schotter, A. "Power Relationships in the International Monetary Fund: The Consequences of Quota Changes," manuscript in progress.

Lucas, W. "Measuring Power in Weighted Voting Systems," Technical Report #227, School of Operations Research and Industrial Engineering, Cornell University, September, 1974.

Raanan, J. "The Inevitability of the Paradox of New Members," Technical Report #311, School of Operations Research and Industrial Engineering, Cornell University, September, 1976.