Core Allocations and Competitive Equilibrium — A Survey*

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Introduction

The following paper is a survey of much of the work done on the game theoretical analysis of competitive equilibrium. While it is, of course, impossible to touch upon every article written on this subject, the following paper does mention most of the major contributions to the field. My principal aim here is one of exposition. I shall claim no credit for the originality of any of the proofs below, and in most cases I shall follow the original proofs of the authors mentioned quite closely, so that the reader can easily refer to them for further reference.

Section I defines, very generally, the concept of the core in relation to the other game theoretical solution concepts.

Section II gives an intuitive explanation of why the game theoretical approach to problems of competitive equilibrium should be considered an advance over the more conventional Walrasian approach.

Section III offers a set of formal definitions found in the literature on Game Theory, along with a formal definition of the core.

Section IV starts a review of the literature on the core and competitive equilibrium, and motivates the discussion by referring to Edgeworth's (1881) book.

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Section V briefly explains Shubik's (1959) contribution to the subject.

Section VI reviews the work of Scarf and Debreu and the proof of three theorems which relate the core to the competitive equilibrium and prove that in the limit they coincide.

Section VII discusses the work of Aumann and others, who analyze markets with a continuum of traders.

Section VIII describes the work of Shubik and Shapley on markets with non-convex preferences among the traders, and comments on the notion of an $\varepsilon$-core.

Section IX will describe the generalization of models of exchange to include production.

Finally, Section X will discuss the recent efforts to extend the concept of the core to economies with public goods. Several criticisms of this effort are offered along with an alternative formulation of the characteristic function for such situations.

I. What is the Core?

The core is one of many solution concepts that have been developed for $n$-person games since The Theory of Games and Economic Behavior was written. While the term is attributed to the work of Gillies [18] and Shapley [49], the concept is clearly visible in Von Neumann and Morgenstern's analysis of three person market games. Economics aside, the core has no special place among the other solution concepts to $n$-person cooperative games, and is in some ways far more unrealistic in its assumptions about communications possibilities than are such concepts as $\psi$-Stability [27], Bargaining Set [36], or the Von Neumann-Morgenstern Solution [59].

The reason why the core has caught the attention of economists is mainly due to Shubik's illustration that the problem discussed in Edgeworth's model of exchange is equivalent to an $n$-person game whose solution coincides with the core and to Scarf's proof that as the number of consumers in an economy becomes large, the set of imputations in the core approaches the competitive allocation.

Unfortunately, however, the great attention paid to the core has done much to hamper the application of other game theoretical solution concepts to problems in economics. The work of Shapley and Shubik is a noted exception, of course.
II. The Relation of the Core to the Theory of General Equilibrium: An Intuitive Explanation

Before I describe the literature on the core, it might be advisable to indicate precisely why the concept of the core, and the game theoretical analysis of competitive equilibrium, should be considered an advance over the conventional approach.

The game theoretical analysis of competitive equilibrium accomplishes two very important things. First, it deduces and defines prices instead of merely assuming their existence\(^1\). Secondly, it is better capable of incorporating the existing institutional framework of an economy into an analysis of prices and allocation than is the neoclassical approach\(^2\). Let us consider these one at a time.

The parametric role of prices is one of the cornerstones of the theory of general equilibrium. Prices simply exist, and a tâtonnement process is invoked which (along with the necessary conditions needed for fixed points) assures us that eventually an equilibrium price vector will be reached. The problem is, of course, that it is the role of economics to explain the existence of prices, and not to assume them.

In the game theoretical analysis, however, the parametric role of prices is not assumed. Rather, prices are the result of large scale multi-person bargaining. The theory says the following: Instead of prices being assumed at the outset, let the traders in the economy have perfect communication possibilities at zero cost. Therefore, we assume that any possible coalition of traders can form and allocate their initial resources among themselves. If, at the end of this very involved process, no trader or group of traders can do better for themselves than to form one “Grand Coalition” of all traders, and allow a certain set of allocations determined there to be the ones they accept as final, the core exists. If not, the core is empty (i.e., there does not exist any imputation with the above properties) and we must rely on some other solution concept to make sense out of the situation. Game theory has several such “other” solution concepts, e.g. the ε-core (see [55], p. 812), the Nash-equilibrium for non-cooperative situations, the Shapley value (see [44]), ψ-stability for situations where coalition formation is restricted, etc.\(^3\)

To relate this to the neo-classical theory, we can say that what happens is that all of the bargainers realize that the lengthy bargain-

\(^1\) See O. Morgenstern (33) who has emphasized this point quite emphatically.

\(^2\) See L. Shapley and M. Shubik (52).

\(^3\) For a good description of some of the relationships between these terms see L. Shapley and M. Shubik (46).
The process is then as follows: If the economy is "essential" (i.e., if the tastes and initial endowments in the economy are such that trade is beneficial) the traders in the economy will commence large scale multi-lateral bargaining in which all possible coalitions of traders can constitute themselves and distribute its goods. If the conditions for a non-empty core exist (Section VI, Assumptions A. 1—A. 4) the traders will eventually realize that the coalition of all traders, the "Grand Coalition", can determine a set of allocations that cannot be dominated by any trader or group of traders. This set is then the set of allocations in the core. Finally, it can be demonstrated (Sec. VI) that to each allocation in the core we can relate a price vector which would have led the traders in the economy to this allocation if that price vector had initially been called out and all traders had acted as price takers.

Consequently, in the game theoretical analysis, the existence of prices is logically deduced and not merely assumed. The set of prices that characterize the allocations of the core, turn out to be (in the limit) identical to the set of competitive prices in the neo-classical analysis.

Why, then, can the game theoretical analysis more easily incorporate the institutional framework of an economy into its analysis? The answer is simple. Theoretical economics considers itself to be an institutionally free science, i.e., a purely theoretical discipline. However, if it assumes the parametric role of prices, it must assume the existence of markets, and since markets are indeed institutions, economics cannot be institutionally free.

There are two ways to get around this problem. One is to introduce institutions directly into the analysis. The second is to deduce the existence of institutions from the theory. The theory of the core does the latter, for if we look close enough we will recognize that if the core exists, the "Grand Coalition" described above is in reality the "market" referred to so frequently in economic texts.

Consequently, this criticism is not valid for the game theoretical analysis since it deduces the existence of markets from the analysis itself and is therefore a more "pure" concept. In addition, if any of

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4 L. Shubik and M. Shapley have often commented on this point.
the necessary requirements for a non-empty core fail to exist, and the core is empty, the Theory of Games can supply us with a variety of solution concepts which, when coupled with the given institutional framework of the economy, can determine equilibrium payoff configurations with far more accuracy than the typical neo-classical theory permits.\(^5\)

III. Formal Definitions\(^6\)

**Definition 1:** A Characteristic Function in a game with transferable utility is a function which maps each subset of players ($S$) in an $n$-person game, into the set of real numbers (the set of cardinal utilities).

**Definition 2:** An imputation is a utility vector, $x = (x_1, x_2, \ldots, x_n)$ with the following properties:

$$x_i \geq u_i,$$

where $u_i$ is the utility level that player $i$ can guarantee himself by playing alone.

$$\sum_{i=1}^{N} x_i = V(N),$$

where $V(N)$ is the total utility created by the formation of the “Grand Coalition”.

**Definition 3:** In a game with transferable utility, an imputation $z = (z_1, z_2, z_3, \ldots, z_n)$ is blocked by a coalition $S$ if $S$ is an effective coalition and the characteristic function maps $S$ into a cardinal utility index whose value is greater than the sum of utilities for all $i$ in $S$ under $z$. More formally, an imputation $z = (z_1, z_2, \ldots, z_n)$ is blocked by $S$ if $\sum_{i \in S} z_i \leq V(S)$.

**Definition 4:** The Core is the set of all imputations that cannot be blocked by any subset of players.

As defined above the characteristic function of a game is a mapping from the set of subsets $S$ of $N$, to the real line (the set of cardinal utilities).

\(^5\) Another advantage of the core concept is that while the competitive equilibrium will exist only if all of the assumptions of perfect competition hold, the core of a market may exist in extremely small markets where traders are far from price takers. For a good analysis of these situations see L. Telser (56), Chapter 1.

\(^6\) For a thorough glossary of game theoretical terms see, G. Schwödiauer (44).
utility indexes). An underlying assumption here is that utility is transferable so that the properties of imputations i.e. blocking, effectiveness, etc., depend only on the sum of utilities and not on the utilities themselves. In economies, however, we may have transferable money, but not utility. This is so if we specify non-linear utility functions for the players. In this context, the concepts defined before must be modified, but as Aumann [4] has demonstrated, the same solution concepts can easily be derived.

Consider the set of players in a game (market game) and any arbitrary coalition \( S \). Define an \( n \)-dimensional Euclidean utility space whose dimension is equal to the number of players, and whose coordinates have as subscripts the players in \( N \). \( E^S \) will be a subspace of \( E^n \) whose dimension is equal to the number of players in \( S \), and whose coordinates have as subscripts the players in \( S \). The characteristic function of this game will associate with each subset of players \( S \), a set \( V_s \) in \( E^S \), which represents the set of possible utility vectors that can be achieved by that coalition. The vector will differ according to the activities engaged in by the members of \( S \). Consequently, the characteristic function has changed from a function which associates a real number with each coalition, and then requires the coalition to agree upon its distribution to one which determines a set of utility vectors represented by the points in \( E^S \).

For models of exchange this difference becomes clear. Take an exchange economy with \( N \) players and \( N \) utility functions (one for each player), and \( N \) vectors \( w_i \) of initial resources. A vector \( u \in E^S \) will be said to be in \( V_s \) if we can find commodity bundles \( x_i \) with \( \sum x_i = \sum w_i \) and \( u(x_i) \geq u_i \) for all \( i \) in \( S \) and where \( u_i \) is the initial utility of player \( i \) before entering the market.

For a game with transferable utility in which a number \( V(S) \) is associated with each coalition, the equivalent definition would be that \( u \in E^S \) may be obtained by \( S \) if \( \sum_{i \in S} u(x_i) \leq V(S) \).

To define the core in these circumstances, we simply let \( u \) be a point in \( V_n \) and \( u^* \) its projection onto \( E^S \). Then, \( u \) is blocked by the coalition \( S \) if there exists a \( y \in V_s \) with \( y \geq u^* \). This means that there exists a vector \( y \) in the set \( V_s \) defined by the characteristic function that gives the members of \( S \) more than \( u^* \) (the projection of \( u \) onto \( E^S \)). A point \( u \in V_n \) is in the core if it cannot be blocked.

To conclude, the concept of a core for market games with non-transferable utility concerns itself with proving the existence of a set of trades without the aid of transferable utility, that satisfy the core constraints. The existence of this set of trades for certain mar-
kets, can be proven using the concept of a “balanced game” and a “balanced set”\(^7\). In addition, all of the models described below do not assume the existence of transferable utility.

**IV. Literature on the Core**

The roots of the theory of the core can be found in Edgeworth’s 1881 book *Mathematical Psychics*\(^8\). In addition to the introduction of indifference curves and discussion of bilateral exchange, Edgeworth asserts that in a two commodity world, if the number of identical traders on both sides of the market become “large”, the contract curve will asymptotically approach the competitive equilibrium.

In the light of the Theory of Games, this assertion takes on new significance. In a two trader world, the core is easily proven to be the “contract curve”\(^9\). This is evident since the only three possible coalitions are the coalition of all traders and the coalitions of single traders acting alone. Since all points on the “contract curve” dominate the no trade situation, all of core constraints are instantly satisfied.

\(^7\) Definition: *A Balanced Set*: Let \( T \) be a collection of coalitions \((S)\). \( T \) is called a balanced set if there is a set of non-negative numbers \( \delta \) for every \( S \) in \( T \) such that the following condition is satisfied:

\[
\sum_{i \in S} \delta_i = 1 \text{ for all } SCN, \text{ with } S \neq N,
\]

where \( N \) is the set of all players.

Definition: *A Balanced Game*: Let a game with \( N \) players and transferable utility have a characteristic function \( V(S) \) defined for all possible coalitions. The game is called a balanced game if for every balanced collection,

\[
\sum_{s} \delta_s V(s) \leq V(N)
\]

(Definitions from Telser (56) 1971, pp. 71—72).

Many authors have used the concept of a balanced game to prove that the core of a game is non-empty (see L. Telser 1971, p. 72, and L. Shapley 1967, pp. 453—460). H. Scarf (41) treats exchange economies that have no money and strictly ordinal utilities and proves that convex preferences among traders implies a balanced market game exists which has a non-empty core. For a good intuitive explanation of the concept of a balanced set and a balanced game, see L. Telser (56, pp. 68—94) and H. Scarf (41), and for an explanation of games without transferable utility, see R. J. Aumann (4).

\(^8\) While it is usually agreed that Edgeworth had first posed the basic problem, a closer look at C. Menger (30) 1871 will indicate that much of what Edgeworth was later to discuss can be found there. O. Morgenstern points this out in (33).

\(^9\) It is also the Von Neumann-Morgenstern solution (59).
Now the “contract curve” or core contains many points including the competitive equilibrium. Edgeworth’s theorem states that as \( n \) becomes large, more and more points in the core can be blocked by intermediate coalitions, until ultimately only one, the competitive equilibrium, is left\(^{10}\). The process that achieves this result is recontracting, which stipulates that no trades need take place if any trader or group of traders can do better. Recontracting is, of course, identical with the game theoretical concept of “Blocking”.

V. Shubik 1959 (53)

Edgeworth’s theorem seemed doomed to a life of obscurity, especially in the Anglo-American literature, until Shubik ([53], 1959) reinterpreted his analysis using the tools of Game Theory\(^{11}\). It was this paper that opened the door to the modern game theoretical analysis of competitive equilibrium. Shubik’s main result is summarized in his

**Theorem 1**: “An Edgeworth market game \((N, N)\) for any size \(N\), where the number of traders in one commodity is the same as the number of traders in the other commodity, and traders have the same preferences, will have a solution consisting of all imputations of the form

\[
[2P\psi(a/2, b/2) - P\psi(0, b) + (1 - P)\psi(a, 0), \ldots, 2(1 - P)\psi(a/2, b/2) + P\psi(0, b) - (1 - P)\psi(a, 0), \ldots],
\]

\(0 \leq P \leq 1\) ([53], p. 271).

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\(^{10}\) For a good and elementary explanation of this “blocking” process see Vivian Walsh Introduction to Contemporary Microeconomics, New York, 1970, pp. 175—176. See also M. Shubik and L. Shapley (52, pp. 72—74).

\(^{11}\) The emphasis here is placed on the words “Anglo-American” for it seems as if some Austrian economists studying in Vienna in the 20’s and 30’s were very well aware of the importance of Edgeworth’s contribution. For an insight into how much impact Edgeworth did indeed have, see the obituary of Edgeworth written by O. Morgenstern in 1926 and published in 1927 in Zeitschrift für Volkswirtschaft und Sozialpolitik 5, No. 10—12, pp. 646—652.

In addition, an equivalent analysis to Edgeworth’s can be seen in the Theory of Games and Economic Behavior, for the case of bilateral exchange, pp. 555—564. Their “common-sense” solution for a three person market game is equivalent to the core of that simple market. For a good discussion of the core and its relationship to questions in partial equilibrium analysis, see L. Telser: Competition, Collusion, and Game Theory. (56).
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In this imputation, all traders on the same side of the market obtain equal amounts. The set of imputations can be characterized by the parameter $P$, which can be interpreted as the market price. This set of imputations is again a “contract curve”, this time for an $(N, N)$ Edgeworth market game. No proof is given as to whether or not the set of imputations in the core tends to converge to the competitive equilibrium. Instead, Shubik discusses several instances in which the core can be proven to exist, and one in which it does not. This seminal paper has been expanded in several ways by recent writers. The main thrust of this research can be divided into six main categories.

1) The proof of Edgeworth’s theorem of the asymptotic behavior of the core (Scarf [43], Scarf and Debreu [14]).

2) The expansion of the analysis to include $m$ types of traders with $r$ traders of each type (Scarf [43], Scarf and Debreu [14]).

3) The expansion of the cardinality of the set of traders to coincide with the continuum (Aumann [3], Vind [58]).

4) The analysis of Cores for economies with non-convex preferences (Shapley and Shubik [55]).

5) The inclusion of production into the analysis. (Scarf and Debreu [14]).

6) The treatment of economies with public goods, (Foley [16], Mileron [41]).

We will study these in order.

VI. The Asymptotic Behavior of the Core: An Exchange Economy with $m$ types of Traders and $r$ Traders of each Type — Scarf (43), Scarf and Debreu (14)

The first mathematical proof of Edgeworth’s theorem was given by Scarf [43]. This article generalizes the Edgeworth-Shubik analysis and covers an economy with many goods, large but finite types of traders, and finite numbers of traders of each type. While this article was a great advance, the proofs given were long and difficult. Fortunately, this shortcoming was rectified by Scarf and Debreu [14]12. It is this paper that we will follow.

12 Actually, Debreu offered a simplification of H. Scarf’s proof in (12), but their joint article is even easier to follow. More recently, H. Scarf and G. Debreu have written another paper (40) which has even greater clarity.
A. Schotter:

Consider a pure exchange economy $(E)$. Assume there are $m$ consumers each with a complete pre-ordering over the points in the non-negative orthant $\Omega^+$ of the commodity space $\mathbb{R}^n$. Scarf and Debreu make the following assumptions about the preferences of consumers:

**A:1 Insatiability:** Let $x$ be an arbitrary non-negative commodity bundle. We assume that there exists a commodity bundle $x'$, such that $x' \succ_i x$.

**A:2 Strong Convexity:** Let $x'$ and $x$ be arbitrary different commodity bundles with $x' \succ_i x$, and let $\alpha$ be a number such that $0 < \alpha < 1$. Then $\alpha (x') + 1 - \sum_i (x_i') \succ_i x$.

**A:3 Continuity:** We assume that for any non-negative $x'$, the two sets $\{x / x' \succ_i x'\}$ and $\{x / x_i \preceq x'\}$ are closed.

If each consumer owns an initial commodity bundle $w_i$, then the initial endowment of the economy can be represented by $n m$-tuples $W = (w_1, \ldots, w_m)$.

**A:4 Strict Positivity of Individual Resources:** This means that each trader owns a strictly positive quantity of each good. In an economy such as this, the final result of trading consists of a collection of $m$ non-negative commodity bundles $X = (x_1, \ldots, x_m)$ which satisfy the feasibility constraint $\sum_{i=1}^m (x_i - w_i) = 0$, which means that the economy cannot allocate more commodities than it has in its possession.

**Definition 5:** Let $(x_1, x_2, \ldots, x_m)$ with $\sum_{i=1}^m (x_i - w_i) = 0$, be an assignment of the total supply to the various traders, and let $S$ be an arbitrary set of traders. We say that an allocation is blocked if we can find at least one other commodity bundle $x'_i$ for all $i \in S$, such that $\sum_{i \in S} (x_i' - w_i) = 0$, and $x'_i \succeq x_i$ for all $i \in S$.

**Definition 6:** The core is the set of all allocations of the total supply that cannot be blocked.

This definition relies totally on ordinal utilities, and does not require the use of a transferable utility.

Scarf and Debreu now prove that as the number of traders becomes infinite, the allocations of the core approach those of the competitive allocation. Another way of saying this is that as the number of traders increases, only the allocations in the competitive allocation do not get blocked.
They employ the following strategy:

1) Prove Theorem 2: Given assumptions A:1—A:4, a competitive equilibrium exists.

2) Prove Theorem 3: The competitive equilibrium is in the core.

3) Prove Theorem 4: If there are $m$ types of traders in an economy, and $r$ traders of each type, and $(x_1, x_2, \ldots, x_m)$ is in the core for all $r$, then it is a competitive allocation.

We shall follow their steps.

**Theorem 2:** The existence of a competitive equilibrium:

Debreu [10] has proven that the four assumptions stated above are sufficient conditions for the existence of a competitive equilibrium. In essence, this is equivalent to proving that there exists a non-negative commodity array $(x_1, x_2, \ldots, x_m)$, with $\sum_{i=1}^{m} (x_i - w_i) = 0$, and a price vector $p$, such that $x_i$ satisfies the preferences of the $i$-th consumer, subject to his budget constraint $p x_i \leq p w_i$.

**Theorem 3:** The competitive allocation is in the core.

**Proof:** This proof is rather straightforward. First, notice that if $x$ is a competitive allocation, then $x_i' > x_i \Rightarrow p x_i' > p w_i$. If this were not true, then $x$ does not satisfy the budget constraint $p x_i \leq p w_i$, for the $i$-th consumer. In addition, $x_i' \geq x_i \Rightarrow p x_i' \geq p w_i$ by assumption A:1 & A:2. Let $S$ be a possible blocking set so that $\sum_{i \in S} (x_i' - w_i) = 0$, with $x_i' \geq x_i$ for all $i \in S$, and strict preference for at least one. From the two remarks made above, if $x_i' \geq x_i$ for all $i \in S$, with strict preferences for at least one, then $p x_i' \geq p w_i$ for all $i \in S$, with strict inequality for at least one. Therefore, $\sum_{i \in S} p x_i' > \sum_{i \in S} p w_i$, a contradiction of $\sum_{i \in S} (x_i - w_i) = 0$.

More intuitively, this theorem says that the only allocations that can block a competitive allocation, violate the feasibility constraint $\sum_{i \in S} (x_i - w_i) = 0$. No trader or group of traders can block a competitive allocation and still satisfy the feasibility constraint.

Finally, to prove Edgeworth's theorem, Scarf and Debreu study an economy with an ever increasing number of traders. Imagine an economy with $m$ types of trader and $r$ traders of each type. By the same type we mean traders who have identical preferences and initial endowments. The economy, therefore, consists of $mr$ traders and
we can index each by an ordered pair \((i, q), (i = 1 \ldots m), (q = 1 \ldots r)\). The feasibility conditions now become,

\[
\sum_{i=1}^{m} \sum_{q=1}^{r} (x_{iq} - r \sum_{i=1}^{m} w_i) = 0.
\]

To simplify matters, Scarf and Debreu prove the following theorem which we will simply assert.

Assertion 1: An allocation in the core assigns the same consumption to all traders of the same type.\(^{13}\)

This assertion allows us to describe the allocations in the Core by \(m\) non-negative commodity bundles instead of \(mr\), with the feasibility condition simplified to its original form \(\sum_{i=1}^{m} (x_i - w_i) = 0\).

The particular bundles in the core will of course depend on \(r\), but we can see that the core for \(r + 1\) traders of each type, is contained in the core for \(r\).

This is obviously true since the economy with \(r\) members can be considered merely as a coalition in the economy with \(r + 1\) members and we know by definition of the core that it cannot block an allocation in the core of \(r + 1\).

The general theme of Theorem 4 is that if we consider a competitive allocation in an economy consisting of one trader of each type, and repeat the allocation when we enlarge the economy to \(r\) participants of each type, the resulting allocation is competitive for the larger economy, and consequently in the core. What we wish to prove is that no other allocation is in the core for all \(r\).

Theorem 4: If \(x = (x_1, x_2, x_3, \ldots, x_m)\) is in the core for all \(r\), then it is a competitive equilibrium.

This theorem is far deeper than Theorem 3. Many versions of it have appeared since Scarf and Debreu’s article (see Arrow and Hahn [1]), but their proof remains the most succinct and straightforward. Briefly, the proof involves the construction of a set \(I_i\) for each \(i\), which is equal to the set of all \(z\) such that \(z + w_i \geq x_i\), where \(x_i\) is a member of the core allocation \(x\). \(I_i\) is defined as the convex

\(^{13}\) Recently J. Green (19) has shown that this “equal treatment theorem” holds only if what he calls the ‘strong super-additivity condition’ holds. Basically, this condition says that the “equal treatment theorem” holds if the points in the core cannot be attained by the individual actions of two disjoint subeconomies. Green points out that the strong super-additivity condition is rarely violated.
hull of the union of \( \Gamma_i \). The Minkowski theorem is then applied using a separating hyperplane through the origin \( p \cdot z = 0 \), with normal \( p \), to prove that all allocations that dominate those of the core, for \( n \) sufficiently large, must violate the budget constraint of some consumer at prices \( p \).

**Proof:** Let \( \Gamma_1 \) be the set of all \( z \) such that \( z + w_i > x_i \), and let \( \Gamma \) be the convex hull of the union of sets \( \Gamma_i \). Since for every \( i \), \( \Gamma_i \) is convex (and non-empty), \( \Gamma \) consists of the set of all vectors \( z \) which may be written as \( \Sigma \alpha_i z_i \), with \( \alpha_i \geq 0 \), \( \Sigma \alpha_i = 1 \), and \( z + w_i > x_i \) (i.e. the convex combination of all vectors in \( U \Gamma_i \)).

The following diagram is offered by Scarf and Debreu to help explain the situation for a two trader, two commodity world.

![Fig. 1]

In this diagram we see that \( \Gamma_1 \) is formed by taking all commodity bundles \( x_1' \) preferred to \( x_1 \), and subtracting \( w_1 \) from them. \( \Gamma_2 \) is formed in a similar way. \( \Gamma \) is simply the convex hull of the union of these two sets.

The next step is crucial and is to prove that the origin is not contained in \( \Gamma \). We assert this here, and refer the reader to ([14], p. 43) for its proof. This assertion, coupled with the fact that \( \Gamma \) is convex by virtue of its being a convex hull, and the Minkowski Theorem, establish the existence of a hyperplane through the origin with normal \( p \), such that \( p \cdot z \geq 0 \) for all \( z \in \Gamma \). Therefore, if \( x' \succeq x \), then \( x_1' - w_1 \) is in \( \Gamma_1 \), and hence in \( \Gamma \). This implies \( px_i' \geq pw_i \), where \( pw_i \) is the budget constraint of the \( i \)-th buyer. Since, by insatiablety of preferences, in every neighborhood of \( x_i \) there are consumptions strictly preferred to \( x_i \), we also obtain \( px_i \geq pw_i \). But \( \sum_{i=1}^{m} (px_i - pw_i) = 0 \) must be satisfied by any final allocation. Therefore, \( px_i = pw_i \) for all
$i$, and for all $x_i' \succeq x_i$, we must have strict inequality in $p x_i \geq p w_i$. It is important to realize that this would not always be true if the origin were contained in $\Gamma$, and it is for this reason why the above assertion is so important.

At this point however, Scarf and Debreu note that the proof is not complete. While it has been proven that there exists a price vector ($p$) for all $x$ in the core such that any allocation preferred to it must violate the budget constraint of some buyer, we have not yet proven that $x$ actually satisfies the preferences for all $i$ subject to their budget constraint, i.e. $x_i' \succeq x_i$ actually implies $p x_i' > p w_i$. This is not difficult however. Since $w_i$ is strictly positive, there exists a non-negative $x^0$ strictly below the budget hyperplane. If for some $x_i''$, both $x_i'' \succeq x_i$ and $p x_i'' = p w_i$, the points on the segment $[x^0, x'']$ close enough to $x''$ would be strictly preferred to $x_i$, and strictly below the budget hyperplane. This is a contradiction of assumption A:1. This completes the proof.

To summarize, the proof implies that for any allocation in the core, we can find a price vector compatible with it such that any allocation preferred to it will violate the budget constraint of some buyer. The asymptotic qualities of the proof enter in proving that the origin is not included in $\Gamma$.

VII. Markets with a Continuum of Traders — Aumann (3)

Markets with a continuum of traders were first analyzed by R. Aumann [3], K. Vind [58], and W. Hildenbrand [23], although games with a continuum of traders were studied by Shapley [47], Milnor [32], and Davis [11]. The study of such markets is the logical conclusion of proving limit theorems for finite markets.

The assumption of a continuum of traders may seem unnatural at first, since any economy, while often very large, is always finite. Despite this however, the concept offers two distinct advantages. First as Scarf has pointed out [43], in proving limit theorems for finite economies, it would be far more advantageous to assume an infinity of traders at the outset, since the process of enlarging finite economies involves the comparison of cores for basically different economies, i.e. it is hard to compare the cores of two economies if they contain different numbers of traders. In addition, as Aumann points out [3], the notion of a perfectly competitive economy can only be mathematically modeled by markets in which the traders form a continuum, for it is only in these markets that the effects of traders on market prices is truly negligible.
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Mathematical Properties of Such Models:

Models of markets with a continuum of traders require the use of Lebesgue measurements and integration found in that field of mathematics known as “analysis”. Since these tools are not usually found in the tool box of economists, we will not dwell on this point, but simply note that the concept of integration seen below is not the same as Riemann integration found in calculus.

Since the set of traders now forms a continuum \( T \), represented by the number of points in the closed interval \([0, 1]\), some of our original definitions will have to be altered.

Definition 7: An allocation is an assignment \( x \) for which \( \int x = \int \mu \), where \( T \) is the set of all players and is represented by the points in \([0, 1]\), and where \( \int \mu \) is the initial endowment of the players.

Definition 8: A coalition of traders is a Lebesgue measurable subset \( S \) of \( T \). If it has the measure zero, it is the null set.

Definition 9: An allocation \( y \) dominates an allocation \( x \) via coalition \( S \) if \( y(t) \succeq x(t) \) for all \( t \in S \), and \( S \) is effective for \( y \) i.e. \( \int_S y = \int_S \mu \).

These definitions are all we need to construct a characteristic function for a market game with a continuum of traders. In such markets, in which the number of traders is infinite to begin with, the proof of Edgeworth’s theorem is made more intuitive. This is true since we no longer need to prove a limit theorem, but only have to prove that for these markets, the set of imputations in the core and those of the competitive allocation are the same. This result is only true for markets with a continuum of traders since the limit theorems for finite economies only prove that as \( n \) approaches infinity, the set of allocations in the core approaches the competitive allocation, but this does not rule out the existence of allocations in the core in addition to the competitive ones.

We will not prove Aumann’s theorem, since despite his use of measure theory, the strategy of the proof is strikingly similar to the one discussed in the finite case above. All that can be said in conclusion is that the analysis of markets with a continuum of traders seems to be the ultimate step in modeling a perfectly competitive economy, and as Aumann has said [3], any shortcoming it may have (especially the assumption of perfect competition) is also evident in the finite model.

Three recent notes on atomless economies by D. Schmeidler [45], Birgit Grodal [20], and K. Vind [58] all comment on the process of blocking for such economies. As we have stated before, an
allocation is in the core of an economy if it cannot be blocked by any coalition of traders or agents. In this process, all possible coalitions must be checked before we can conclude that the core is non-empty. However, in atomless economies, Schmeidler has proven that if an allocation is blocked by a coalition \( S \), then there is also a blocking subcoalition of arbitrarily small size (or measure) \( \xi \). Consequently, in atomless economies we need only check those coalitions with measure less than \( \xi \) to see if an allocation is in the core.

In addition, in atomless economies, if only coalitions with measure less than \( \xi \) are allowed to form, we will still have an identity between the allocations in the core and the set of Walras equilibria.

Finally, a new mathematics developed by A. Robinson [28] called Non-Standard Analysis has just recently begun to be applied to atomless economies by D. Brown and A. Robinson [7, 8]. Since I cannot pretend to understand this new endeavor, I can only offer you the words of the authors themselves [6]. While discussing the two existing treatments of proving the equivalence between the core allocations and the competitive equilibrium, the authors state:

One approach has been to talk about a sequence of economies growing without bound, and to look at the relationship between the core and the set of competitive equilibria for very large economies. This was the method of Debreu-Scarf.

The other approach has been to consider an exchange economy having an infinite number of traders, to define the notions of core and competitive equilibrium in this economy, and to show the equivalence between these two concepts. Aumann’s work on continuous economies has been of this nature.

Here, we report the results obtained by a new method for the resolution of Edgeworth’s conjecture, based on non-standard analysis, which synthesizes the asymptotic method of Debreu-Scarf and the infinite method of Aumann. We have shown that within nonstandard analysis the concept of the core and competitive equilibrium are the same. As a consequence of this theorem, we have derived a number of asymptotic results concerned with unbounded families of standard exchange economies. (6, 1258).

VIII. Cores and Non-Convex Preferences — Shapley and Shubik (55)

While the existence of a competitive equilibrium is doubtful in economies where the traders have non-convex preferences, this does not mean that these economies should not be studied. Shapley and Shubik’s paper [55] demonstrates how far we can go in this direction.

The main theorem of this paper is that in exchange economies with non-convex preferences among the traders, there exist certain
“quasi-cores”, which will be defined later, that exhibit at least “sociologically” stable properties, but which do not clear all markets. While the existence of such “quasi-cores” has been proven by the authors, they point out that they have not as yet found conditions to prove that as the number of traders becomes infinite, these “quasi-cores” approach the core.

To begin, Shapley and Shubik (S. & S.) make two rather restrictive assumptions about the traders in the economy. First they assume the existence of “money” in addition to goods and assert that this money possesses a constant marginal utility for each trader represented by a constant $\lambda_i$ which enters the utility function in an additive fashion. Therefore, the utility function of each trader can be represented as follows

$$u^i (x_1^i, x_2^i, \ldots, x_m^i, M) = U^i (x_1^i, x_2^i, \ldots, x_m^i) + \lambda_i M.$$ 

Second, they assume that with the proper normalization, all utility functions for the traders can be represented by one general function. This is equivalent to assuming equal tastes among the traders.

With these assumptions, S. & S. define two “quasi-cores” for a market game in characteristic function form with non-convex preferences among the buyers.

**Definition 9:** A strong $\varepsilon$-core is a set of pay-off vectors satisfying

$$\sum_{i \in S} \alpha_i \geq V(S) - \varepsilon_1 \text{ for all } S \subseteq N.$$ 

**Definition 10:** A weak $\lambda$-core is a set of pay-off vectors satisfying

$$\sum_{i \in S} \alpha_i \geq V(S) - s \varepsilon, \text{ where } s \text{ is the number of players in } S.$$ 

These “quasi-cores” then state than an imputation can be blocked by any coalition $S$ only if that coalition offers each of their members a certain profit over and above that which is dictated by the usual concept of blocking. More intuitively, these numbers, $\varepsilon$ represent the costs of coalition formation, and these costs are determined by the institutional framework of the economy. Obviously, in the strong $\varepsilon$-core, these costs are fixed for any size coalition, while for the weak $\varepsilon$-core, the costs are variable and depend on the size of the coalition.

Once again, we will not prove S. & S’s theorems about the existence of such $\varepsilon$-cores in markets with non-convex preferences, since such proofs impart more mathematical than economic information.
The best way to conclude, therefore, is to simply quote from the conclusion of S. & S.:

"Another tool that seems potentially useful in this connection (analyzing markets with non-convex preferences) is the quasi-core as developed in this paper. It can easily be shown that $\varepsilon$-cores (both weak and strong) always exist if $\varepsilon$ is large enough. The sociological factor involved here can be interpreted as an orginational cost prerequisite to cooperative action, proportional to the parameter $\varepsilon$. Theorems 2 and 4 of this paper indicate that even if $\varepsilon$ is small, the quasi-core will exist if the market is large enough. Of course, when $\varepsilon$-cores exist for small values of $\varepsilon$, it is not unlikely that the core itself exists as well, making the market fully stable against recontracting. But even without a true core, the profit to be gained from recontracting out of an $\varepsilon$-core would be small and near stability can be achieved ([55], p. 823).

IX. The Core of a Productive Economy — Scarf and Debreu (14)

While we have previously employed the concept of the core to characterize exchange economies, it is easily generalized to describe productive economies. This generalization was first accomplished by Scarf and Debreu [14].

Assume that production takes place in an economy whose technology can be described by a production set $Y$. Each coalition of consumers is assumed to have access to this same set. A point $y \in Y$ represents a production plan which can be represented by a vector of commodities with positive components for outputs, and negative components for inputs. Scarf and Debreu assume the following: $A.5$ — $Y$ is a convex cone with vertex at zero.

It is interesting to note (as Scarf and Debreu do) that a pure exchange economy is merely a special case of a productive economy in which $Y$ is degenerate and contains only the origin.

An allocation for this economy is a collection of non-negative commodity bundles $(x_1, x_2, \ldots, x_m)$ such that there exists a $y \in Y$ satisfying the condition,

$$\sum_{i=1}^{m} x_i = y + \sum_{i=1}^{m} w_i \text{ i. e. } \sum_{i=1}^{m} (x_i - w_i)$$

belongs to $Y$.

More simply this means that the economy can produce all of those goods that cannot be supplied by the initial endowment. This allocation is blocked by a coalition $S$ if it is possible to find commod-
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ity bundles \( x_i' \) for all \( i \in S \) such that \( \sum_i (x_i - \omega_i) \) belongs to \( Y \) and \( x_i' \succeq x_i \) for all \( i \) in \( S \), with strict preference for at least one. The core consists of all those allocations that cannot be blocked by any \( S \).

With these relatively minor changes, the proofs of theorems 2—4 can be given along almost identical lines as before.

Following the same procedures as before, we will use the following strategy:

1) Prove Theorem 5: If \( x = (x_1, \ldots, x_n) \) is a competitive equilibrium for a productive economy, it is in the core.

2) Prove Theorem 6: If \( x = (x_1, \ldots, x_n) \) is in the core for all \( r \), then it is a competitive equilibrium.

Proof of Theorem 5: As Scarf and Debreu [14] point out, Assumptions A:1—theorems of Section VI and A:5 are no longer sufficient to prove that a competitive equilibrium exists. However, if a competitive equilibrium exists, we will now prove that it is in the core.

This is a simple extension of Theorem 3. Let \( x \) be a competitive allocation, and let \( S \) be any possible blocking coalition which proposes \( x' \) as its blocking allocation. By definition \( \sum_{i \in S} (x_i' - \omega_i) = y \in Y \) with \( x_i' \geq x_i \) for all \( i \) in \( S \) with strict preferences at least once. Also, \( p y \leq 0 \).

However, since \( p x_i' \geq p \omega_i \) for all \( i \) in \( S \), with strict inequality for at least one \( i \), we have \( \sum_{i \in S} p x_i > \sum_{i \in S} p \omega_i \). Since by definition \( p y = \sum_{i \in S} p (x_i' - \omega_i) \), we know that \( p y \geq 0 \). A contradiction.

Assertion: An allocation in the core assigns the same consumption to all consumers of the type.

Proof of Theorem 7: If \( x = (x_1, \ldots, x_n) \) is in the core for all \( r \), then it is a competitive equilibrium.

This theorem is proven in exactly the same way as was done in the case of a barter economy. However, since we now have a production set \( Y \), which is larger than a single point (the origin) we must adjust our reasoning appropriately to take this into consideration.

Let \( \Gamma' \) be defined as before to be the convex hull of the union of \( m \) sets \( \Gamma_i = \{ z \mid z_i - \omega_i \geq x_i \} \). Again, \( \Gamma' \) can be considered the set of points \( z \) that agent \( i \) prefers to \( x_i - \omega_i \). Let \( Y \) (the production set) be a convex cone with vertex at the origin. First we must show that \( \Gamma' \) and \( Y \) are disjoint. This is done by an argument similar to the one used in the proof of Theorem 4 in which it was proven that the
origin does not belong to \( I \). (See Scarf and Debreu [14] for the full proof.) Consequently, if the two sets \( I \) and \( Y \) are disjoint and convex, they may be separated by a hyperplane through the origin with normal \( p \) such that \( pz \geq 0 \) for all \( z \) in \( I \), and \( py \leq 0 \) for all \( y \) in \( Y \). Consequently, we can proceed as we did in the case of a barter economy to show that for any allocation in the core there exists a price vector, \( p \), such that any allocation preferred to it is not feasible i. e. \( py > 0 \). More intuitively, we have proven the existence of a hyperplane through the origin that separates \( I \) and \( Y \). All points in \( Y \) are below \( py = 0 \), and all points in \( I \) are above \( py = 0 \). The only points they have in common are the allocations in the core which are in the closure of \( I \). Therefore at prices \( p \), all allocations preferred to the core allocations are strictly above the hyperplane and consequently not in \( Y \) and not feasible\(^{14}\).

The final step is again to show that \( x = (x_1, \ldots, x_n) \) actually satisfies the preferences of the consumers. The proof of this is similar to that of Theorem 3 in Section VI, and we refer the reader to that section for its proof.

X. Cores for Economies with Public Goods — Foley (16)

One of the recent attempts in the application of Game Theory to the study of competitive equilibrium has been the application of the theory of the core to economies with public goods. In my opinion, however, this attempt has fallen far short of a valid theory of public competitive equilibrium. The basic problem here is that the core solution is inappropriate for the purpose at hand if the present interpretations of the characteristic function and the blocking process are used. As D. Foley [16] has said:

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\(^{14}\) T. Hansen and J. Gabszewicz (22) introduce into a productive economy the possibility of collusive agreements among factor owners, in which restrictions are placed upon the process of coalition formation in which factor owners band together and agree that:

a) no proper subset of them will accept to enter a coalition with agents outside of their group, but only the group as a whole will enter such a coalition.

b) The total amount of the output made available to the group after production will be shared evenly.

In economies where such collusive agreements exist, Hansen and Gabszewicz prove that the core exists and is stable for that modified economy. However, in such economies the equal treatment theorems may have to be modified in many instances.
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The difficulty in the public goods economy is that the definition of blocking used in Section 5 makes blocking difficult because a dissenting coalition must produce its own public goods and lose the benefits of the externalities generated by the rest of the economy. As a result the core, even as the number of traders becomes large includes many allocations. Further thought on this subject may produce either a better definition of blocking or a clearer characterization of the value properties of points in the core. (16, p. 72).

On the other hand, the core concept may be retained if an alternative “probabilistic” form of the characteristic function suggested below is employed or if the game is described in terms of Robert Rosenthal’s [60] concept of a game in “Effectiveness Function Form”.

We will proceed as follows: First we will discuss the concepts of a public competitive equilibrium and a Lindahl Equilibrium. Then, following Duncan Foley’s original article, we will show that if an allocation is a Lindahl equilibrium it is in the core. Finally, we will discuss some possible criticisms of this approach and offer an alternative formulation.

Foley’s Model of Public Competitive Equilibrium

Let E be a productive economy with m public goods, k private goods and n agents. In addition, let $X^i$ be a closed convex consumption set for agent $i$ which has an interior in the private goods subspace. Each agent is assumed to have continuous, convex, and monotone preferences over $X^i$, and the economy consumption set $X = \sum_{i=1}^{n} X^i$ is bounded from below for $\leq$. The production set, $Y$, is a closed convex cone with vertex at zero which can produce any public good, although public goods are assumed not to be necessary for production.

A vector of public and private goods can then be written as $(x_1, \ldots, x_m; y_1, \ldots, y_k) = (x; y)$. In this economy the following definitions hold:

1) An allocation is a vector of public goods $x \in E^m$ and a set of $n$ vectors of private goods $(y_1, \ldots, y^n)$ each in $E^k$ such that for all $i$ there is $(x; \bar{y}^i) \in X_i$ with $\bar{y}^i < y^i$.

2) A Feasible Allocation is then an allocation $(x; y_1, \ldots, y^n)$ such that $(x; \sum_{i=1}^{n} (y^i - w^i) \in Y)$.

18 In this section we will change notation slightly and denote traders by superscripts instead of subscripts.
3) A Pareto Optimum — is a feasible allocation \((x; y^1 \ldots y^n)\) such that there is no other feasible allocation \((\bar{x}; \bar{y}^1 \ldots \bar{y}^n)\) with \((\bar{x}; \bar{y})_i > (x; y)\) for all \(i\).

4) A public Competitive equilibrium is a feasible allocation \((x; y^1 \ldots y^n)\), a price system \(p = (p_x; p_y)\), and a vector of taxes \((t^1 \ldots t^n)\) with \(p_x x = \sum_{i=1}^{n} t_i\) (i.e. a self financing public sector) such that a) \(p \cdot (x; \sum_{i=1}^{n} (y^i - w^i)) \geq p \cdot (\bar{x}; \bar{z})\) for all \((\bar{x}; \bar{z}) \in Y\), i.e. the allocation maximizes the profits of the producers; b) If \(p_y \cdot y^i = p_y \cdot w^i - t^i\) and if \((\bar{x}; \bar{y}^i) \succeq_i (x; y^i)\) then \(p_y \cdot \bar{y}^i - t^i > p_y \cdot y^i\), i.e. the allocation satisfies the preferences of the consumers. c) There is no vector of public goods and taxes \((\bar{x}; \bar{t}^1 \ldots \bar{t}^n)\) with \(p_x \bar{x} = \sum_{i=1}^{n} \bar{t}_i\), such that for every \(i\) there exists a \(\bar{y}^i\) with \((\bar{x}; \bar{y}^i) \succeq_i (x; y^i)\) and \((p_y \cdot \bar{y}^i) \prec (p_y \cdot w^i - \bar{t}^i)\) i.e. the competitive allocation is a Pareto Optimum.

As Foley points out, in this type of equilibrium each consumer is charged a different price for the public goods and that price is changed until he demands an equal amount of the public good as all other consumers\(^{16}\).

A Lindahl Equilibrium differs from this public competitive equilibrium in that while a public competitive equilibrium admits the existence of transfer payments between agents, the Lindahl Equilibrium does not. More specifically, if \(L^i = p_x^i x + p_y y^i - p_y w^i\) is a lump sum transfer to agent \(i\), the Lindahl equilibrium constrains the allocation process to satisfy \(L^i = p_x^i x + p_y y^i - p_y w^i = 0\) for all \(i\).

Therefore, a Lindahl Equilibrium must be defined with relation to a fixed income for each agent.

5) A Lindahl Equilibrium with respect to \(w = (w^1 \ldots w^n)\) is a feasible allocation \((x; y^1 \ldots y^n)\) and a price system \((p_x^1 \ldots p_x^n; p_y) \geq 0\) such that

\[a) \left[ \sum_{i=1}^{n} p_x^i; p_y \right] \cdot [x; \sum_{i=1}^{n} (y^i - w^i)] \geq \left[ \sum_{i=1}^{n} p_x^i; p_y \right] - (\bar{x}; \bar{z})\] for all \((\bar{x}; \bar{z}) \in Y\),

\[b) \text{if} \ (\bar{x}^i; \bar{y}^i) \succeq_i (x^i; y^i) \text{ then} \ p_x^i \bar{x} + p_y \bar{y}^i > p_x^i x + p_y y^i = p_y w^i.\]

Using this formulation, Foley proves that a Lindahl equilibrium exists. Again we will not prove this theorem because it is so similar to the ones given previously in Sections VI and IX of this paper. The strategy

\(^{16}\) The classical references here are of course E. Lindahl (26), P. A. Samuelson (37), (38), R. Musgrave (62) and L. Johansen (25).
of his proof is interesting however, and this does merit some discussion. Basically, Foley takes the private goods economy developed by Scarf and Debreu [14] and defined in the k dimensional private goods space $E^k$, and extends it into an $nm+k$ dimensional space by defining each of the n consumer’s allocations of the m public goods as a separate goods. Therefore, in this space each consumer will have $n-1-m$ zeros in the place of those public goods not concerning him. The sets $I'$ and $Y$ of Sections VI and IX are then defined in this space and are proven to be convex and disjoint. Finally, as in Section IX, the existence of a separating hyperplane with normal $p$ is proven to exist and the existence of the Lindahl Equilibrium is then proven along the lines of Scarf and Debreu [14].

This being accomplished, the following theorem can be proven:

**Theorem 8:** If $(x; y^1, \ldots, y^n), (p_{x^1}, \ldots, p_{x^n}; p_y)$ is a Lindahl equilibrium with respect to $w$, it is in the core with respect to $w$.

**Proof:** Suppose, given the Lindahl equilibrium $(x; y^1, \ldots, y^n)$, coalition $S$ proposes $(\bar{x}; \bar{y}^1, \ldots, \bar{y}^n)$ as a blocking allocation. Since $(\bar{x}; \bar{y}^i) \succ_i (x; y^i)$ for all $i \in S$ we know by definition of a Lindahl equilibrium that

$$\sum_{i \in S} p_{x^i} \bar{x} + p_y \sum_{i \in S} \bar{y}^i > \sum_{i \in S} p_{x^i} x + p_y \sum_{i \in S} y^i = p_y w$$

i.e. the Lindahl equilibrium satisfies the preferences of each consumer. Since $p_{x^i} \geq 0$ for all $i$, $\sum_{i=1}^n p_{x^i} \geq \sum_{i=1}^n p_{x^i}$ which means that since $\bar{x} \geq 0$, $\sum_{i=1}^n p_{x^i} \bar{x} + p_y \sum_{i=1}^n (\bar{y}^i - w^i) > 0$. But the profit maximizing condition for a Lindahl Equilibrium asserts that $\sum_{i=1}^n p_{x^i} \bar{x} + p_y \bar{z} \leq 0$ for all $(\bar{x}; \bar{z}) \in Y$. This contradiction proves the theorem. As Foley points out, it is interesting to notice that it has not been proven that as an economy becomes infinite, the allocations in the core shrink asymptotically and approach the Lindahl Equilibrium. What this means is that in the limit, the set of core allocations is larger than the set of Lindahl Equilibria. This problem exists, I feel, because an inappropriate definition of the characteristic function and blocking is employed in the usual formulation of the problem.

More intuitively, what Theorem 8 says is that there exists a vector of public and private goods and a set of prices and taxes such that any coalition cannot do better for its members than to accept the prescribed allocation. However, this form of the characteristic function requires that a blocking coalition provide the public goods for itself if it protests the allocation prescribed by society. In essence,
under the present formulation of the game, a coalition protests its
treatment under the present tax system and threatens not to pay,
saying that it can provide at least an equivalent vector of private
goods for itself and possibly a better vector of public goods. How-
ever, for truly public goods (in the sense of Samuelson [37]), such
a threat is meaningless. It fails to take into account the fact that any
public goods provided by a coalition are also provided free of charge
to the rest of the economy and vice versa. In addition, it fails to
specify the reaction of the rest of society to this threat.

A more realistic blocking strategy for a coalition would be for it
to disguise its preferences and to say that it did not care enough
about the public goods under consideration to pay for them at the
prescribed prices. If this seems as the relevant threat, then the character-
istic function must be redefined as follows:

The value of the characteristic function in a market game with
transferable utility, for a coalition (S) protesting an allocation and a
price structure in a public goods economy, is equal to the value of the
private goods vector that it can insure for all of its members at
100% probability, plus the expected value of the amount of the
public goods the rest of society (−S) can be expected to supply with-
out it. More specifically, if \( V(S)_\text{public} \) is the value to coalition \( S \) if
coalition \( −S \) (or society minus \( S \)) provides the public goods vector \( x \) at quantity \( q \), then the expected value of the public goods to coaliti-
on \( S \) if it refuses to pay for them is equal to

\[
E \left[ V(S) \right]_{\text{public}} = \int_0^\infty V(S)_q \, f(q) \, dq,
\]

where \( f(q) \) is a density function defined over the quantity of the
public good supplied by \((−S)\).

Therefore, the value of the characteristic function for a coalition
\( S \) if it protests the prescribed tax structure is equal to

\[
V(S) = V(S)_{\text{private}} + E \left[ V(S) \right]_{\text{public}}
\]

\[
= V(S)_{\text{private}} + \int_0^\infty V(S)_q \, f(q) \, dq_{17}.
\]

\text{17} The "probabilistic" characteristic function form of the game is a
special case of the "Effectiveness Function Form" of the game defined by
R. Rosenthal (60). Rosenthal objects to certain uses of the characteristic
function for games on the grounds that it does not recognize the conditional
nature of threats in a game. For an excellent example of the type of situation
that Rosenthal has in mind, see (60), Example Two.

In the Effectiveness Function Form of the game, each coalition can
restrict the game to a certain range and the final outcome then becomes
In this context, an allocation \((x; y)\) is blocked by coalition \(S\) if 
\(V(S) = V(S)^{\text{private}} + E[V(S)]^{\text{public}} \geq V[s(x^t) + y^t]\) and the core will be non-empty if we find an allocation \((x; y)\) such that 
\[V[s(x^t) + \sum_{i \in S} y^t] \geq V(S)^{\text{private}} + E[V(S)]^{\text{public}}\]
for all \(S \subseteq N\) and 
\(U^i(x^t + y^t) \geq U^i(\nu^t + EV(i)^{\text{public}})\) for all \(i \in N\)^{18}.

This definition of the characteristic function is a “probabilistic” one in which each coalition acting alone gets the best possible allocation it can for itself by producing and distributing its private goods optimally, and then calculating an expected externality using a density function \(f(q)\).

As an aside to this discussion, it is interesting to note that if the goods under consideration are club goods (i.e., goods equally available to all members of the club if they are not excluded by the membership requirements), then the core solution and the usual concepts of the characteristic function and blocking can be left intact. As M. Pauly [61] points out, if a coalition in a club protest its imputation, it can threaten to form a new club with other players both inside and outside of the present club, and provide the goods for itself. In this case, the existence of a core depends upon the economies of scale of club good production and the relationship of the optimal size of a club to the total population of the economy. This case however, is quite distinct from the case of a pure public good, even though the usual formulation fails to notice the distinction.

Many interesting aspects of collective action can be explained using this formulation. To begin, as M. Olson has pointed out [35], small groups tend more frequently to provide public goods voluntarily than do large groups. From the above formulation we can see that this is obviously true since in a small group each coalition is relatively “important” in that it constitutes a large percentage of the total membership of the group. Consequently, it realizes that without its cooperation large quantities of the public good cannot be supplied. More specifically, the coalition is forced to realize that the rest of society, acting alone, cannot supply a large expected externality and as a result the value of the characteristic function for this coalition is small because \(E(V(S))^{\text{public}}\) is small. Consequently, blocking be-

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^{18} For ease of exposition we have defined this characteristic function as if a transferable unit of utility existed. It is of course easy to generalize from this to the case of non-transferable utility.
comes difficult for any coalition and the core will most likely exist and be large. Ironically, the larger a coalition is in society, the weaker its bargaining power is, vis à vis the provision of public goods, since it becomes obvious that it cannot rely on the rest of society to provide the goods without it.

Conversely, a small coalition in a large economy can easily block a proposed allocation since it knows that it can rely on the other members of the economy to provide large quantities of the public goods without it (E[V(S)]^public is extremely large).

Even more striking is the power of individual agents in economies with public goods. Here voluntary contribution would be impossible to rely on since an individual can depend, with almost 100% probability, that the total amount of the public goods will be supplied in its absence. Blocking in this situation becomes exceedingly easy, and the core exceedingly small, if it exists.

Whether or not the core exists will depend on the preferences of the agents and on the technology that exists to produce public goods. Intuitively, however, it seems unlikely that public goods will ever be provided voluntarily in “large” economies. Some form of governmental coercion will always be necessary. Consequently, a totally cooperative solution, such as the core, for such a game seems unlikely.

Whether the concepts defined above can be applied efficiently to the problem of a public competitive equilibrium or a Lindahl Equilibrium is still an open question. However, the formulation presented, I feel, is a step in the right direction in that it presents a more realistic description of the true blocking process that occurs in economies with public goods and public sectors. The fact that any protesting coalition will still benefit from the externality provided by the rest of society when it provides the public goods is clearly introduced, and the relevant threat capabilities of coalitions is described.

XI. Conclusions

As can be seen from this article, the game theoretical analysis of competitive equilibrium has received quite a bit of attention during the past ten years. It is literally impossible to keep up with the multitude of articles and discussion papers written on the subject, and this article does not pretend to have done so. On the contrary, it has endeavored to present the logical progression of complexity that has characterized this subject by paying attention to those articles which have opened new fields of investigation for future authors.

One ironic note on this entire subject, however, is to notice how far removed the present game theoretical analysis is from the original
intensions of the authors of the *Theory of Games and Economic Behavior*. To be more precise, Von Neumann and Morgenstern in the introduction to their book, protest the formulation of a competitive economy as a collection of isolated maximizing agents whose only function is to calculate maximum consumption and input-output vectors, given fixed parameters which are beyond their control. In the authors words:

Let us look more closely at the type of economy which is represented by the “Robinson Crusoe” model, that is, an economy of an isolated single person or otherwise organized under a single will...

Thus Crusoe faces an ordinary maximum problem the difficulties of which are of a purely technical...

Consider now a participant in a social exchange economy. His problem has, of course, many elements in common with a maximum problem. But also it contains some very essential elements of an entirely different nature. He too tries to obtain an optimum result. But in order to achieve this he must enter into relations with others... Thus each person tries to maximize a function... of which he does not control all variables. This certainly is no maximum problem but a peculiar and disconcerting mixture of several conflicting maximum problems. (59, p. 10).

More recently, O. Morgenstern has written (see [33]) criticizing the common use of the word ‘competitive’ in economic theory. He says:

Consider “competition”: the common sense meaning is one of struggle with others of fight, of attempting to get ahead or at least to hold one’s place. It suffices to consult any dictionary of any language to find that it describes rivalry, fight, struggle, etc. Why this word should be used in economic theory in a way that contradicts ordinary language is difficult to see. No reasonable case can be made for this absurd usage which may confuse and must repel any intelligent novice.

In current equilibrium theory there is nothing of this true kind of competition: there are only individuals firms or consumers, facing given prices fixed conditions, each firm or consumer for convience insignificantly small and having no influence whatsoever upon the existing conditions of the market (rather mysteriously formed by tatonnement) and therefore solely concerned with maximizing sure utility or profit... (33, p. 1164).

However, the present game theoretical treatment of General Equilibrium is also guilty of this lack of competitiveness. While the process leading to the establishment of the core is indeed a process of rivalry, the eventual solution is purely cooperative and can be approximated by a set of competitive prices. This is no surprise, in the sense that if we give two correct theories the same set of assumptions, they will arrive at the same conclusions. However, it certainly can not be considered the type of solution that Von Neumann and Morgenstern
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had in mind when they wrote the Theory of Games and Economic Behavior.

If we are to infer anything from that book it must be that rivalry and bluffing is not to stop when a solution is reached, but that these threats and fights must form the basis for the eventual stability of the system. The final solution envisioned there was obviously not a totally cooperative equilibrium, but one in which each set of coalitions or individuals holds the others in check by pooling their available strategies.

The concept of the core is indeed a fascinating idea. However, the rules of the game necessary for its existence (i.e. free and perfect communication, zero costs to coalition formation, etc.) are so removed from reality that the final result of the analysis obviously reveals all of the faults attributed to the neo-classical analysis. It is for this reason that I feel that the overwhelming emphasis on the core has done a disservice to the game theoretical analysis of economic situations. The true value of Game Theory lies not in the analysis of idealized special cases, but rather in its ability to analyze more common though less appealing (at least from a theorist’s point of view) real life economic phenomena.

Appendix

The following chart is included to aid the student who is interested in furthering his understanding of this difficult but fascinating subject. It is meant to be a self-contained list of readings which in-

<table>
<thead>
<tr>
<th>GAME THEORETICAL BACKGROUND</th>
<th>NEO-CLASSICAL BACKGROUND</th>
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Core Allocations and Competitive Equilibrium — A Survey


4 Francis Y. Edgeworth, Mathematical Psychics, London 1881.


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includes all of the necessary Game Theoretical and Economic literature. One point must be emphasized, however, and that is that while this list is self-contained, it is by no means complete. I have chosen only the articles which are most essential and easiest to understand from each of the topics covered in this paper, but many more certainly do exist, as the bibliography demonstrates. The list is meant to be read in order, but if the reader has already read certain material, or is familiar with certain concepts, he should certainly skip them. I hope that this will facilitate the reader’s grasp of this subject.

References


Core Allocations and Competitive Equilibrium — A Survey


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