Equilibrium in a durable goods market with lumpy adjustment

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Abstract

Durable goods are an important component of the business cycle. Equilibrium models of durable goods markets are made difficult by the lumpy nature of individual purchases. We show that a straightforward approximation of the distribution of durable goods holdings gives rise to a tractable equilibrium model. We analyze the case of competition as well as that of a monopoly producer.

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1. Introduction

The consumption of durable goods, like investment, is a volatile component of aggregate demand that has proved difficult to model successfully. One problem is that consumers generally purchase durable goods, such as automobiles and furniture, infrequently. Fluctuations in aggregate demand therefore reflect both fluctuations in the number of agents making purchases, and fluctuations in the demand of any single agent.
The classical microeconomic model of discrete and infrequent adjustment is the \((S,s)\) model of Arrow et al. [4]. \(^1\) The problem with \((S,s)\) behavior is that it is notoriously difficult to aggregate, since the cross-sectional distribution of agents’ holdings becomes a state variable. This distribution determines how many agents are near their purchase triggers and therefore short-run demand. To handle the high dimension of this distribution, current models either ignore equilibrium altogether, limit the analysis to special cases, or resort to highly stylized computational methods.

The importance of solving for equilibria is highlighted by several recent empirical studies. Parker [34] finds that durable goods prices rise in response to demand shocks, while Bils and Klenow [11] show that durable goods prices tend to move procyclically relative to non-durable goods prices. This suggests that endogenous price movements may be important and that consumers may take price movements into account in their adjustment decisions. Adda and Cooper [2] present more striking evidence. They estimate a structural model of discrete purchases on U.S. and French automobile data and find that the endogenous response of agents’ purchases to price movements is very important in understanding the response of auto sales to an income shock, in fact more important than the evolution of the distribution holdings. \(^2\)

The goal of this paper is to construct a workable model of the market for a durable good that incorporates infrequent purchases. Our approach is to simplify the equilibrium dynamics by abstracting from the echoes of previous cycles. High demand today creates a lump in the cross-sectional distribution of holdings at “big \(S\).” As the lump passes through the \((S,s)\) bands it is smoothed out by heterogeneity in depreciation, tastes, income and wealth. Incomplete smoothing produces an echo in demand as the lump reaches “little \(s\).” This echo creates a link between the market today and the market in the far future. Breaking this link greatly simplifies the analysis.

In this paper, we assume that there is sufficient heterogeneity across agents in the time between purchases that the echoes effectively disappear. This implies that the cross-sectional density of durable goods holdings is fairly flat in the neighborhood of the adjustment trigger. The resulting model is very flexible, and we use it to analyze the equilibrium dynamics of price and the number and size of purchases in response to a variety of shocks and in a variety of market structures.

While our primary motivation is methodological, our model has a number of interesting properties. One important finding is that, if marginal costs are increasing or producers have monopoly power, then durables sales look more like the AR(1) found in the data than the ARIMA\((0,1,1)\) predicted by Mankiw’s [32] theoretical model. The reason is that endogenous price movements tend to spread fluctuations in demand over time. As a result there is a trade-off between the size and persistence of fluctuations. If marginal costs are flat, demand fluctuations will be sharp and short-lived. If marginal costs are rapidly increasing, they will be longlasting and low amplitude.

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\(^1\) Grossman and Laroque [28] apply the \((S,s)\) model to the theory of an individual’s consumption of a durable good, and Eberly [25] and Attanasio [3] test its empirical implications on panel data.

\(^2\) In order to solve their model, Adda and Cooper assume that the price process is exogenous to agents’ decisions to purchase autos. Their estimation, however, suggests that causation goes both ways. In order to fit the data they must assume that their price shock is correlated with their demand shock.
A second important result is that markups are naturally procyclical in response to demand shocks and counter-cyclical in response to cost shocks. This is because the ability of consumers to time purchases places limits on producers’ monopoly power. One implication is that cost movements must be sufficiently large and procyclical, in order for markups to be counter-cyclical on average.

An important issue concerns the accuracy of the approximation. We provide two “tests” of the assumptions. First, we show theoretically that if the time between purchases is sufficiently long, then echoes disappear. The assumptions therefore work in principle. Second, we follow Krussel and Smith [30] and assume that agents base their expectations about price on some moments of the cross-sectional density of holdings. We then calibrate the model to aspects of the U.S. automobile market and simulate it. In these simulations, the echoes effectively disappear. We find that agents’ forecasts do not deteriorate significantly if they ignore fluctuations in the density of holdings near the purchase trigger. The assumptions therefore also appear to work in practice.

The next section places the paper in the context of the literature on (S,s) aggregation. We present the model in Section 3 and introduce the approximation in Section 4. We solve for the equilibrium of our simplified model in Section 5. In Section 6, we present a linearization of the model and analyze its properties. Section 7 extends the model to the case of a monopoly producer. Section 8 contains the theoretical and empirical justification for the main assumptions of the model. Section 9 concludes.

2. Literature review

In recent years, a large body of research has developed to examine the relationship between microeconomic frictions and aggregate dynamics. Three general responses to the difficulty of placing (S,s) models in equilibrium settings have emerged. First, many authors have made great progress in understanding the aggregate dynamics by ignoring equilibrium issues altogether. Work by Blinder [13], Bertola and Caballero [9], Caballero [15], Eberly [25], Carroll and Dunn [21], and Adda and Cooper [1] proceeds by assuming that shocks to an individual’s desired consumption are exogenous, and that each individual agent follows an (S,s) policy consistent with the dynamic properties of these shocks and a fixed cost of purchasing the durable. Aggregates result from summing over individual behavior given some assumption about the correlation of shocks across individuals. This literature focuses

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3 The (S,s) approach is not the only approach to modelling durables. Leahy and Zeira [31] obtain an analytic model of infrequent purchases by assuming that agents purchase the durable only once in their lifetime. Their model is best suited to large purchases such as a home. There are also several equilibrium models with representative agent consumers, most notably Blanchard and Melino [12], Murphy et al. [33], and Baxter [6]. The representative agent provides a microfoundation for the consumption of durable goods, but not one that accords with observed microeconomic behavior.

4 In some of these models agents choose the (S,s) triggers optimally. In others the authors choose the policy in order to fit the data. Carroll and Dunn include a precautionary savings motive. Adda and Cooper solve for a competitive equilibrium but with constant marginal cost, so price is effectively exogenous. In both Carroll and Dunn and Adda and Cooper the adjustment triggers may shift in response to exogenous shocks.
on the dynamics of the distribution of agents’ holdings within the (S,s) bands, not on the
equilibrium determination of the bands themselves.

The second branch of the (S,s) literature searches for settings which simplify the distribu-
tional dynamics, making it possible to include equilibrium considerations. Caplin and
Spulber [20], Benabou [7,8], and Caplin and Leahy [18,19] construct (S,s) models of pric-
ing in which the cross-sectional distribution of relative prices is uniform. This allows them
to construct equilibrium models and, in the case of Caplin and Leahy [19], a model with
aggregate dynamics. In none of these models, however, does the timing or magnitude of the
adjustment respond to the state of the cycle. The bands respond to changes in the parameters
of the model, but they are constant through time.

The final branch of the literature makes assumptions that reduce the size of the state space,
making possible computational solutions in the spirit of real business cycle models. Fisher
and Hornstein [26] solve an (S,s) inventory model by discretizing firms’ inventory holdings.
Dotsey, King and Wolman [24] solve a model of pricing by considering a setting in which
there are only a small number of different prices at any point in time. Following Krussel
and Smith [30] a number of papers consider various approximations of the distribution of
holdings within the (S,S) bands. While these approaches are all promising, it is also useful
for the purpose of developing our intuition to have simple analytic models of durable goods
cycles.

The approach taken in this paper is a cross between the second and the third outlined
above. As in the third branch, we show that our approximation yields dynamics that are a
first-order approximation of the actual dynamics. As in the second branch, we show that
our assumption implies that the distribution of holdings in the neighborhood of “little s” is
uniform. In our model, however, the (S,s) bands adjust in response to shocks. Our approach
is also complementary to the first class of models. Where those papers fix the policies and
focus on the distributional dynamics, we, in a sense, fix the distribution and focus on the
equilibrium determination of the policies.

3. Modeling durables

Our goal is to introduce equilibrium considerations into a model of infrequent adjustment.
We focus on the direct effects of supply and demand on market equilibrium and make
assumptions that limit the indirect effects of durable purchases through wealth and capital
accumulation.

We begin with a general model of a market for a durable good and show how the cross-
sectional distribution of holdings makes solving for an equilibrium difficult. In the next
section we present our main simplifying assumption: the absence of echo effects.

3.1. A general model

Consider a competitive market for a durable good. Time is discrete and indexed by $t \geq 0$.
There is a continuum of consumers indexed by $i$ who are identical except that at any given
date $t$ they may possess different quantities of a durable good. Let $K_{it} > 0$ denote the
quantity of the durable held by agent $i$ at time $t$, and let $\varphi_i(K)$ denote the cumulative
distribution of agents’ holdings of the durable good at date $t$. The distribution of holdings will evolve as durables depreciate and as agents exchange old durables for new ones.

Durables depreciate geometrically on average. In order to introduce some heterogeneity into the model, we assume that there is some randomness in the depreciation process:\(^5\)

$$K_{it} = \xi_{it} K_{i,t-1}$$  \hspace{1cm} (3.1)

where the random variable $\xi_{it}$ is independent across time and across agents, and is continuously distributed over $[a, 1-b]$ for some $a, b \in (0,1)$. Let $1 - \Delta$ denote the mean of $\xi$ so that $\Delta$ is the average rate of depreciation. The important properties are: (i) geometric depreciation which preserves the homogeneity of the value function; (ii) some heterogeneity which contributes to the mixing of the distribution of holdings;\(^6\) and (iii) some minimal amount of depreciation, so that agents who are indifferent between adjusting today and adjusting tomorrow will strictly prefer to adjust tomorrow if they wait.

Consumers only hold one durable at a time. Their main decision is when to scrap their old durable and purchase a new one. Since scrapping is a type of fixed cost of adjustment, agents will purchase durables infrequently, allowing each purchase to depreciate for some time before making a new purchase.\(^7\)

Consumers choose the amount and timing of their expenditures in order to maximize the present value of utility. Each consumer’s utility in each period depends only on the quantity of the durable held, and is additively separable between the consumption of the durable and the consumption of other goods. Let $U(K) = K^\alpha / \alpha$, denote the utility that a consumer derives from $K$ units of the durable in any given period.

Let $V(K_t, \omega_t)$ denote the value of an optimal policy for a consumer holding a durable of size $K_t$ given that the state of the market, to be discussed in detail below, is $\omega_t$. This problem may be written as

$$V(K_{it}, \omega_t) = \max_{\{T_j, S_{T_j}\}} E_t \sum_{s=t}^{\infty} \beta^{s-t} U(K_{is}) - \sum_{j=1}^{\infty} \beta^{T_j-t} p(\omega_{T_j}) \lambda(\omega_{T_j}) S_{T_j},$$ \hspace{1cm} (3.2)

where

$$K_{is} = \begin{cases} K_{it} & \text{if } s = t \text{ and } T_1 > t, \\ S_{T_j} & \text{if } s = T_j, \\ \xi_{is} K_{is-1} & \text{otherwise}. \end{cases}$$

$E_t$ is the mathematical expectation conditional on date $t$ information. The first summation represents the utility that the agent receives from the durable. $\beta$ is the consumer’s discount

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\(^5\)In principle any source of heterogeneity will smooth echoes, including heterogeneity in tastes, local prices, or the marginal utility of wealth. Heterogeneity in depreciation is particularly easy to model, since we will be able to consider one single market price and one single economy-wide marginal utility of wealth.

\(^6\)Here assuming that $\xi$ is i.i.d. works against us. Assuming instead that there is some serial correlation in $\xi$ would cause the distribution of holdings to spread more quickly and the echoes to disappear more quickly.

\(^7\)We assume a scrapping cost for simplicity. We could assume that only a fraction of the durable is scrapped or that the sale price of capital is a fixed fraction of the purchase price, so long as both the adjustment cost and the depreciation rate are great enough that adjustment is always in one direction: towards larger durable holdings.
rate, and \( U(K_s) \) is the utility from holding a durable of size \( K_s \). The second summation represents the cost of successive purchases of the durable. \( T_j \) is the date of the \( j \)th purchase. On these dates the consumer purchases \( S_{T_j} \) units of the durable good at a unit price of \( p_{T_j} \). \( \lambda_{T_j} \) is the marginal utility of wealth and translates the purchase price into utility terms. The total utility cost of each purchase is therefore \( p_{T_j} \lambda_{T_j} S_{T_j} \). Between purchase dates the durable depreciates according to (3.1). Both \( p \) and \( \lambda \) depend on the state of the market, and are exogenous to the individuals’ decision. We discuss the determination of \( p \) and \( \lambda \), as well as their exogeneity, below.

Before moving on, it is important to emphasize two important implications of the assumptions that we have made thus far. First, all agents who purchase a durable in period \( t \) purchase the same quantity. We will denote this quantity \( S(\omega_t) \) in reference to “big S” in the (S,s) model. In general \( S \) will depend on the state of the market \( \omega_t \).

Second, with scrapping agents will only adjust from a smaller durable to a larger one so long as the marginal utility of durable consumption is positive. 8 There will be some cutoff value \( s(\omega_t) \), which will depend on the state of the market \( \omega_t \), such that all agents with \( K_{it} < s(\omega_t) \) will make purchases. Here we have chosen the notation \( s(\omega_t) \) in reference to “little s” in the (S,s) model.

Consider the marginal individual who is indifferent between purchasing a new durable and retaining her existing one. The value to making a purchase is \( V(S_t, \omega_t) - p(\omega_t)\lambda(\omega_t)S_t \). If the shocks are sufficiently small relative to the minimal depreciation rate, the agent will make a purchase the next period, so the value to waiting is \( U(K_t) + E_t[V(S_{t+1}, \omega_{t+1}) - p(\omega_{t+1})\lambda(\omega_{t+1})S_{t+1}] \). The level of \( K \) that makes the agent indifferent is defined by

\[
U(K_t) = V(S_t, \omega_t) - p(\omega_t)\lambda(\omega_t)S_t - E_t[V(S_{t+1}, \omega_{t+1}) - p(\omega_{t+1})\lambda(\omega_{t+1})S_{t+1}].
\]

Since none of the terms on the right-hand side depend on the quality \( K_t \), the adjustment trigger is the same for all agents.

The number of agents that purchase the durable in period \( t \), \( n_t \), will be a function of \( s(\omega_t) \) and the distribution \( \varphi_t \):

\[
n(\omega_t) = \varphi_t (s(\omega_t)).
\]

Given their homogeneity, all consumers who make purchases will purchase the same quantity of the durable good. The mass of durables purchased is therefore \( q_t = n_t S_t \). The dynamics of the distribution of holdings follows immediately from the policies \( S(\omega) \) and \( s(\omega) \).

We now turn to the determination of \( p \) and \( \lambda \). Recall that \( p \) is the price of a unit of the durable; a durable of size \( S \) will cost \( pS \). At this point we assume that the supply side of the market is competitive, so that this price is equal to the marginal cost of production.

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8 Work by Caballero [15] and Eberly [25] suggests that most of the dynamics in the durable goods markets that they study come from the behavior of agents who get rid of a durable to buy a larger durable, which suggests that the one-sided assumption is appropriate. Theoretically, the high depreciation rates of most durables also suggests that the one-sided assumption is appropriate.
Later we consider the problem of a durable goods monopolist. Let $\phi$ denote the marginal cost of production. We assume that $\phi(n_t, S_t, c_t) = \pi(n_t)S_t^\gamma c_t$, where $c$ is a cost shock. This production function allows a separate role for the number and size of durable production. We assume that $\pi'(n) \geq 0$ and $\gamma \geq 0$, so that industry-wide marginal costs are weakly increasing in both the number and the size of the durables produced. We will assume that producers do not hold inventories so that the number and size produced is simply the number and size demanded by consumers.

We assume that there are perfect capital markets so that all agents have the same marginal utility of wealth $\beta_t$. This removes the effect of individual income shocks that may complicate aggregation. Moreover, we assume that purchases of the durable are small relative to total lifetime consumption, so that the marginal utility of wealth is effectively exogenous to the outcomes and decisions in this market. This assumption simplifies the analysis by eliminating the simultaneity between the utility of wealth and durable consumption.

Rather than specifying the rest of the economy in detail, we will specify the dynamics of $c_t$ and $\lambda_t$ directly. In this way we can think of placing this market into a variety of macroeconomic settings by altering the dynamics of cost and the marginal utility of wealth. We want to allow $c$ and $\lambda$ to be non-stationary in order to capture the effects of technological progress and of innovations in permanent income. In what follows we will use bars over variables to denote trend or steady-state values and hats to denote deviations from trend. 9

We assume $c_t = \bar{c}_t + \hat{c}_t$ and $\lambda_t = \bar{\lambda}_t + \hat{\lambda}_t$, where trends $\bar{c}$ and $\bar{\lambda}$ follow random walks with drift in logs, $\bar{c}_{t+1} = \rho_c \eta_{ct+1} \bar{c}_t$ and $\bar{\lambda}_{t+1} = \rho_\lambda \eta_{\lambda t+1} \bar{\lambda}_t$. Here $\eta_c$ and $\eta_\lambda$ are a random shocks, that are independently and identically distributed across time, positive and mean one. Drift $\rho_c$ may reflect technical progress or increasing labor costs, while $\rho_\lambda$ reflects discounting and the steady-state interest rate. The cycles $\hat{c}_t$ and $\hat{\lambda}_t$ follow stationary mean-zero AR(1) processes in logs. 10,11 Note that innovations in these processes may be correlated with each other or with innovations in trend.

The state of the market includes the cyclical and trend components of marginal utility of wealth $\lambda$ and the cost shock $c$, as well as the distribution of agents’ holdings of the durable good $\phi$: $\omega_t = (\phi_t, \hat{\lambda}_t, \bar{\lambda}_t, \hat{c}_t, \bar{c}_t)$. Equilibrium in this market is a fixed point in $p$. Given their expectations of the price process, agents choose decision rules $s(\omega_t)$ and $S(\omega_t)$ to maximize expected utility. These decision rules in conjunction with the distribution of holdings give rise an actual price process. In equilibrium the expected price process must correspond with the actual price process.

While it is easy to define an equilibrium, it is more difficult to solve for one. The primary reason is that the distribution of holdings $\phi$ appears as a state variable. It matters how many people have new durables and how many have old ones, since those with old durables

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9 Our detrending procedure is in effect a Beveridge–Nelson decomposition.
10 It is trivial to extend the analysis to a finite-order ARMA process.
11 It is useful to think of the evolution of the marginal utility of wealth in terms of the Euler equation for non-durables:

$$\lambda_t = \beta R_t E_t \lambda_{t+1}.$$  

Here $R$ is the gross real interest rate in terms of non-durables. Our assumptions on $\lambda$ amount to assuming that $R_t = \bar{R}_t \tilde{R}$, where $\bar{R}$ follows a mean zero AR(1) and $\rho_\lambda = 1/\beta \bar{R}$. 

are more likely to purchase new ones. Changes in the distribution of holdings therefore lead to changes in demand which alter the price process. Since this distribution is potentially of large dimension, it complicates both analytic and computational solutions to the model.

4. A simplified setting

Our aim is to develop a tractable model of durable goods cycles in which the number of durables that agents’ purchase and the size of the durable that agents’ purchase respond to shocks to the economy. In what follows we make two approximations that reduce the number of state variables and greatly simplify the dynamics out of steady state. We argue that both approximations become more accurate, the longer is the time between an individual’s purchases.

4.1. No echo effects

As stated in the introduction, echoes complicate matters by creating links between the state of the market today and the state in the far future. If a large number of consumers were to purchase the durable at date \( t \) and the average time between purchases is \( T \), then it is possible that there will be a large cohort of consumers returning to the market \( T \) years later. A shock that leads to abnormally high purchases today therefore may lead to predictable fluctuations in the distribution of holdings near the adjustment trigger in the future. We refer to such fluctuations as echo effects, since they are echoes of previous disturbances to demand.

We see echo effects in many situations. Most prominently the tendency to have children at a certain age leads to echo effects in population dynamics. The echoes, however, are usually smaller than the original disturbance. Idiosyncratic behavior smooths the echo. Baby booms are followed by baby boomlets.

We assume that for all practical purposes the heterogeneity in depreciation eliminates the echo in durable demand. Any lumps that may appear in the distribution of holdings of the durable good near the target “big S” will be smoothed out before reaching the purchase trigger. This implies that the density of holdings is essentially flat in the neighborhood of the adjustment trigger. We assume that this density is equal to \( \mu \).

**Assumption 1 (No echo effects).** Suppose that agents purchase a new durable in period \( t \) if their holding is less than \( s_t \). Then the density of log holdings is uniform in logs with density \( \mu \) in a neighborhood to the right of \( s_t \).

For the moment we simply adopt Assumption 1 as a property of the equilibrium. It is important to note, however, that several properties of the model make an assumption like this feasible. First, as we shall see below, the homogeneity of the model ensures that the ratio of \( S_t/s_t \) is stationary, so that the range of log holdings has of fixed width. If this were not the case then the distribution of holdings would spread or compress over time and
the density of holdings at the adjustment trigger could not remain constant. \footnote{It is tempting to guess that } Second, as shown in Lemma 2 of Section 8, the assumed depreciation process effectively takes the log distribution of holdings, shifts it to the right and then mixes it. The mixing smooths out the distribution. The constant drift preserves uniformity. Third, as shown in Lemma 3 of Section 8, we need that the fluctuations in \( s_t \) be small relative to the minimal depreciation rate \( b \), so that an agent who is at the adjustment trigger in period \( t \) adjusts with certainty by period \( t + 1 \). This ensures that the uniformity of the distribution is preserved by successive truncations at \( \ln s_t \) in spite of the heterogeneity in depreciation.

Given Assumption 1 and the properties of the model that make it feasible, the number of purchases in period \( t \), \( n_t \), is equal to

\[
  n_t \cong \mu (\ln s_t - \ln s_{t-1} + \delta),
\]

where \( \delta = -\ln(1 - \Delta) \). Depreciation causes the log distribution of holdings to fall on average by an amount \( \delta \). If the adjustment trigger does not change then sales would be equal to \( \mu \delta \). Increases in \( s_t \) mean that more agents make purchases at date \( t \). Increases in \( s_{t-1} \) mean that there are fewer agents left over from the previous period with holdings below \( s_t \). Note that, as a result of Assumption 1 and Lemma 3 of Section 8, the number of purchases depends only on the average depreciation rate.

We show in Section 7 that the approximation becomes a good one if the time between purchases is sufficiently long. We also test the validity of the approximation in simulations of the model.

Assumption 1 rules out effects arising from fluctuations in the density of holdings in the neighborhood of the purchase trigger. It is important to note, however, that another class of distributional dynamics remains, namely the dynamics associated with movements in the purchase trigger \( s \). When \( s \) is below its trend potential sales are abnormally high since the increase in \( s \) will trigger sales. In this sense there is “pent up demand.” When \( s \) is abnormally high sales will tend to be abnormally low in the near future and there is “exhausted demand.” Most of the literature on \((S,s)\) aggregation has focused on the first kind of distributional dynamics (changes in the density at given values of \( \hat{K} \)) and ignored the second (changes in \( s \)). \footnote{Carroll and Dunn [21] analyze cyclical fluctuations in the purchase trigger, but ignore equilibrium. Parker [34] allows for changes in \( s \), but the driving process behind his dynamics is the evolution of \( \varphi \). Caballero and Hammour [17] analyze the cyclical properties of the scrapping margin in a vintage capital model.}

4.2. The unpredictability of the future

Echoes are not the only link between the market today and the market in the far future. If shocks are persistent or if the internal propagation mechanisms in the market are strong, the effects of current conditions may be felt for some time. To simplify matters further we assume that the time between purchases is sufficiently long that these dynamics run
their course, and agents learn little from the current state of the economy about the state of the economy at the time of their next purchase, except for the information that the current state contains about the trend. Specifically, we assume that the distribution of the cyclical variables will converge to the unconditional distribution within the time between purchases. Let \( f_t(\hat{c}_{t+\tau}, \hat{\lambda}_{t+\tau}, \hat{s}_{t+\tau}, \hat{\lambda}_t, \hat{s}_t) \) denote the distribution of \( \hat{c}_t, \hat{\lambda}_t, \) and \( \hat{s}, \tau \) years hence conditional on their period-\( t \) values. Let \( T \) be chosen such that only a negligible percentage of durables are held for fewer than \( T \) periods.  

**Assumption 2 (Unpredictability of the future).** \( f_T(\hat{c}, \hat{\lambda}) \approx f_\infty(\hat{c}, \hat{\lambda}) \).

Together Assumptions 1 and 2 make the value of holding a new durable independent of the current state of the cycle. We can decompose the value of a new durable into two terms. The first is the utility derived from the durable over the period that it is held. This is independent of the state of the cycle because of the separability of the utility function. The second is the value of an optimal policy at the date at which the agent replaces the durable. This is independent of the current state of the cycle only if the state of the economy at that date is independent of current state. Assumptions 1 and 2 ensure this independence.

It is important to note that these assumptions do not imply that the size choice is independent of the current state of the cycle. This choice will still depend on the current price and marginal utility. The assumptions only imply that the value of a given size choice is independent of the cycle. The benefits of a given choice are independent of the cycle, the costs are not.

It is also important to note that there is some conflict between the two assumptions. Increases in heterogeneity tend to improve the approximation in Assumption 1, but may also increase the percentage of agents who hold a durable for short periods of time, thereby reducing the accuracy of Assumption 2.

Let \( \tilde{V}(K, \tilde{c}, \tilde{\lambda}) \) denote the value of a new durable of size \( K \). Given Assumptions 1 and 2, \( \tilde{V} \) is independent of the cycle. We will therefore take \( \tilde{V} \) as given in the analysis that follows.  

The only theoretical properties that we need are presented in the following lemma which summarizes some of the trend properties of the model.

**Lemma 1.** Let \( K^*_t \) denote the optimal purchase size when \( c, \lambda, \) and \( n \) are at their trend values and there are no adjustment costs, and let \( \tilde{S} \) denote the optimal purchase size when \( c, \lambda, \) and \( n \) are at their trend values and there is a scrapping cost. Then:

1. \( \tilde{V} \) is twice differentiable with respect to \( K \) with \( \tilde{V}_1 > 0 \) and \( \tilde{V}_{11} < 0 \) for \( K \) in a neighborhood of \( \tilde{S}_t \).
2. \( \tilde{S} \) and \( \tilde{s} \) are proportionate to \( K^* \).
3. \( p_t = \tilde{p}_t \tilde{p}_t \) where \( \tilde{p}_t = \mu \tilde{K}^t \tilde{c}_t \) and \( \rho_p \eta_p \tilde{p}_{t-1} \) is a non-stationary process with \( \rho_p = \rho_c \frac{1-\gamma}{1-2\gamma} \) and \( \eta_p = \eta_c \frac{1-\gamma}{1-2\gamma} \) and \( \tilde{p}_t = \frac{\tilde{\eta}(n_t)}{\tilde{\rho}^s} \tilde{S}^\gamma \tilde{c}_t \) is a stationary process.

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14 Some durables may be held for fewer periods due to extreme realizations of the depreciation process.

15 For practical purposes \( \tilde{V} \) may be approximated by the non-stochastic steady-state value function or the value function for fixed prices. These approximations will be accurate if the shocks are small.
4. $K_t^* = (1 - (1 - \Delta) \rho_p \rho_{\lambda t}) \tilde{\rho}_{t \lambda t}$.

5. There exists a function $v(K_t / K_t^*)$ such that $\tilde{V}(K_t, \tilde{\omega}_t) / K_t^* = v(K_t / K_t^*)$ and the first order condition (3.3) becomes

$$v(S_t / K_t^*) - \frac{\hat{\beta}(\omega_t \hat{\lambda}(\omega_t)}{\Phi} \frac{S_t}{K_t^*}$$

$$= \frac{1}{\alpha} \left( \frac{K_t}{K_t^*} \right)^{\alpha} + E_t \beta \left( \rho_p \rho_{\lambda t+1} \eta_{\lambda t+1} \right)^{1/1 - \Delta}$$

$$\times \left[ v(S_{t+1} / K_{t+1}^*) - \frac{\hat{\beta}(\omega_{t+1} \hat{\lambda}(\omega_t)}{\Phi} \frac{S_{t+1}}{K_{t+1}^*} \right],$$

where $\Phi = (1 - (1 - \Delta) \beta \rho_p \rho_{\lambda t})$ is a constant.

An appendix contains a proof of the lemma along with the proofs of all other propositions. Here we provide some intuition for the results. We begin with the differentiability of $\tilde{V}$. The indifference condition (3.3) implies that the value function is continuous in $K$ for any realization of the exogenous shocks. The value function, however, may be non-differentiable at points of indifference when a small movement in $K$ shifts the time of adjustment from one date to the next. The diffuseness of the depreciation process ensures that these events are measure zero and that these non-differentiabilities disappear as we integrate over sample paths.

The other results relate to the homogeneity of the value function. Given the form of the utility function it is not surprising that the value function is homogeneous of degree $\alpha$ in some variable. The adjustment cost, the shock processes, and marginal cost were all chosen to preserve this homogeneity. We normalize by $K^*$ since it is easy to derive. Result (4) follows from the first-order condition for optimal choice without frictions. Results (2), (3) and (5) are all direct results of this homogeneity.

### 4.3. Summary of the model

We now summarize the main components of the model. The data of the model are:

- The initial values of the exogenous variables, $\tilde{c}_0, \bar{c}_0, \hat{\lambda}_0, \tilde{\lambda}_0$, and the processes for $\hat{c}_t, \bar{c}_t, \hat{\lambda}_t, \tilde{\lambda}_t$.
- An initial distribution for durable goods holdings which is characterized by a minimum holding $s_{-1}$ and a density of log holdings equal to $\mu$ on $K > s_{-1}$.
- A flow utility $U(K)$ and a value function $V(K, \tilde{\omega})$ which captures the value after purchase of a durable of size $K$ given the trend state $\tilde{\omega}$.
- A cost curve $\phi(n, S, c)$.

### 5. Solution to the model

Given that the distribution of durable goods holdings is uniform in logs near the purchase trigger, it is useful to express equilibrium in terms of log durable holdings. Moreover, given
the homogeneity properties expressed in Lemma 1, we will normalize all variables by the trend frictionless optimum $K^*_t$ in order to make the model stationary. Let $\tilde{S}_t = S_t / K^*_t$ denote the normalized target, and let $k_t = \ln K_t - \ln K^*_t$ denote log holdings of agent $i$ relative to the size of the frictionless optimal purchase. Note, $U(K_i) = K^*_t / \bar{x} = K^*_t e^{x(k_i)} / \bar{x} \equiv K^*_t u(k_i)$, and geometric depreciation implies:

$$k_{i,t} = k_{i,t-1} - \ln \tilde{\zeta}_{it} - \ln K^*_t + \ln K^*_{t-1}.$$

5.1. Demand

Consumers make two decisions: when to purchase the durable and how much to purchase. We begin with the size decision. Given Assumption 1, consumers maximize:

$$V \left( S, \bar{\omega}_t \right) = p_t \lambda_t S.$$

Equivalently, in terms of the normalized value function, the consumer chooses $\tilde{S}_t$ to maximize $v(\tilde{S}_t) - \psi_t \tilde{S}_t$, where $\psi_t$ is the normalized price in utility units, $\psi_t = p_t \lambda_t / K^*_t$. For notational convenience we have used $\psi_t$ to capture cyclical movements in both $p_t$ and $\lambda_t$. Let $\tilde{\delta}(\psi)$ denote the result of this optimization. Given $u''(\cdot) > 0$ and $u'''(\cdot) < 0$, both of which hold in the neighborhood of the optimal purchase according to Lemma 1, we have $\tilde{\delta}'(\psi) < 0$. It is convenient to define $f(\psi) = v(\tilde{\delta}(\psi)) - \tilde{\psi}(\tilde{\delta}(\psi))$. $f$ is the cyclical part of the net value of purchasing the durable. Note that $f'(\psi) < 0$ and $f''(\psi) > 0$.

We now turn to the decision of when to purchase the durable. The optimal policy is a cutoff rule. Consumers make purchases if $K_i < s_t$. Let $\kappa_t = \ln s_t - \ln K^*_t$, then consumers make purchases if $k_{i,t} < \kappa_t$. The consumers with $k_t = \kappa_t$ are indifferent between purchasing the durable in periods $t$ and $t + 1$. The payoff from purchasing in period $t$ is $\tilde{V}(S, \bar{\omega}_t) - p_t \lambda_t S$. The gain to delay is that consumers receive the services of their current durable and purchase a new one in the next period. The indifference condition is therefore

$$\tilde{V}(S_t, \bar{\omega}_t) - p_t \lambda_t S_t = U(s_t) + \beta E_t \left\{ \tilde{V}(S_{t+1}, \bar{\omega}_{t+1}) - p_{t+1} \lambda_{t+1} S_{t+1} \right\}.$$

Let $\tilde{\tau}_t = K^*_t / K^*_{t-1}$. We can use the homogeneity of $\tilde{V}$ to express the indifference in terms of the cyclical variables as follows:

$$f(\psi_t) = u(\kappa_t) + \beta E_t \left\{ \tilde{\tau}_{t+1} f(\psi_{t+1}) \right\}.$$

The left-hand side of the equation captures the amount a purchase today differs from trend, and the right-hand side reflects how much the return to delay differs from trend. $\tilde{\tau}_t$ corrects for shifts in trend between $t$ and $t + 1$.

Given the form of the optimal policy it is easy to calculate the quantity sold in each period. Given the absence of echo effects, the distribution of the $k_i$ is uniform with a lower bound of $\kappa_{t-1}$ at the end of period $t - 1$. All of these $k_i$ depreciate by $\delta$ upon entering period $t$. In addition the shift in the trend causes a further reduction of $\ln \tilde{\tau}_t$. The number of agents that make purchases in period $t$ is therefore

$$n_t = \mu[\kappa_t - \kappa_{t-1} + \delta + \ln \tilde{\tau}_t],$$

where $\mu$ is the density of the uniform distribution. Each agent purchases $S_t$. 

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5.2. Competitive equilibrium

Using property 3 of Lemma 1:

\[ \psi_t = \zeta \pi (n_t) \hat{S}(\psi_t) \hat{\lambda}_t, \]

Equation 5.3

where \( \zeta = (1 - (1 - \Delta)\rho_p \rho_h) / \mu \).

Eqs. (5.2) and (5.3) summarize the model. The state variables in the general model were \( \omega_t = (\phi_t, \hat{\lambda}_t, \tilde{\lambda}_t, \tilde{\epsilon}_t, \hat{\epsilon}_t) \). Assumption 1 has allowed us to replace \( \phi_t \) with \( \kappa_{t-1} \). The innovations in \( \tilde{\lambda}_t \) and \( \tilde{\epsilon}_t \) enter (5.2) and (5.3) through \( \tilde{\epsilon}_t \). Note that \( \hat{\epsilon}_t \) and \( \hat{\lambda}_t \) enter Eqs. (5.2) and (5.3) together as a product. Define \( \hat{e}_t = \hat{\lambda}_t \hat{\epsilon}_t \). The stationary formulation of the model represented in (5.2) and (5.3) has two shocks: an innovation in trend \( \tilde{\epsilon}_t \) and a stationary shock \( \hat{\epsilon}_t \).

Let \( \hat{\omega}_t = (\kappa_{t-1}, \hat{\epsilon}_t, \tilde{\epsilon}_t) \). An equilibrium is a pair of functions \( \psi \) and \( g(\hat{\omega}_t) \) such that \( \psi_t = \psi(\hat{\omega}_t) \) and \( \kappa_t = g(\hat{\omega}_t) \) satisfy (5.2) and (5.3) for all \( t \).

Eqs. (5.2) and (5.3) look very much like a supply curve and a demand curve. As in Fig. 1, place \( \psi_t \) on the y-axis and \( \kappa_t \) on the x-axis, the supply curve (5.3) depends on the past choices \( \kappa_{t-1} \), the cyclical shock \( \hat{\epsilon}_t \), and the shock to trend \( \tilde{\epsilon}_t \). Given the state, the set of \( \psi_t \) for which this equation holds is increasing in \( \kappa_t \). This curve shifts to the right with increases in \( \kappa_{t-1} \), and reductions in \( \hat{\epsilon}_t \) and \( \tilde{\epsilon}_t \). Demand (5.2) depends on future price \( \psi_{t+1} \). Given \( \psi_{t+1} \), the set of \( \psi_t \) for which this equation holds is decreasing in \( \kappa_t \). An increase in \( \psi_{t+1} \) increases demand at any given price, so that the curve shifts to the right. The intersection of the two curves determines the functions \( \psi(\hat{\omega}_t) \) and \( g(\hat{\omega}_t) \).

The existence of an equilibrium follows from standard dynamic programming arguments.

**Proposition 1.** A competitive equilibrium exists.
6. A linearization

A linear example will make the dynamics of the model clear. We log linearize (5.2) and (5.3) around the non-stochastic steady state. For simplicity we assume that there is no drift in \( c \) and \( \lambda \), and that the only shock is the cyclical shock. We will be able to analyze permanent shocks by setting the autoregressive coefficient to one. These equations become:

\[
- s^2 \tilde{s}_t = p \lambda (\tilde{h}_t + \tilde{\lambda}_t) - \beta p \lambda S E_t (\tilde{h}_{t+1} + \tilde{\lambda}_{t+1}),
\]

(6.1)

\[
\tilde{p}_t = \frac{1}{1 + \frac{s}{\varepsilon_S}} \left( \frac{\pi'(n)n}{\pi} \frac{s}{\delta} (\tilde{s}_t - \tilde{s}_{t-1}) + \frac{\gamma}{\varepsilon_S} \tilde{\lambda}_t + \tilde{\varepsilon}_t \right),
\]

(6.2)

where tildes denote percent deviations from steady-state values, all other variables are at their steady-state values, and \( \varepsilon_S = -V''(S)S/V'(S) \). This leads to the following second-order difference equation for \( \tilde{s}_t \):

\[
C_1 \tilde{e}_t - \beta C_1 E_t \tilde{e}_{t+1} = C_2 \tilde{s}_{t-1} - \left[ s^2 + C_2 (1 + \beta) \right] \tilde{s}_t - \beta C_2 E_t \tilde{s}_{t+1},
\]

(6.3)

where \( \tilde{e}_t = \tilde{\lambda}_t + \tilde{\varepsilon}_t \) and \( C_1 \) and \( C_2 \) are positive constants that depend on the parameters of the model.\(^{16}\)

We assume that \( \tilde{e}_t \) follows a stationary autoregressive processes:

\[
\tilde{e}_t = \theta \tilde{e}_{t-1} + \eta_t,
\]

and we look for a solution of the form:

\[
\tilde{s}_t = x \tilde{s}_{t-1} + y \tilde{e}_t.
\]

Matching coefficients:

\[
x = \frac{1 + C_2 (1 + \beta)}{2 \beta C_2} - \frac{1}{\beta} \left[ s^2 + C_2 (1 + \beta) \right]^2 - \frac{1}{\beta} \in [0, 1]
\]

\[
y = \frac{-C_1 (1 - \beta \theta)}{1 + C_2 [1 + \beta (1 - x - \theta)]} < 0.
\]

Given this solution we can calculate the evolution of the number of purchases, \( n \), total sales, \( q = n S \).

\[
\delta \hat{n}_t = s (\tilde{s}_t - \tilde{s}_{t-1}),
\]

\[
\delta \hat{q}_t = \delta \tilde{s}_t + s (\tilde{s}_t - \tilde{s}_{t-1}).
\]

(6.4)

The dynamics of price follows from (6.2).

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\(^{16}\) \( C_1 = \frac{p \lambda S}{1 - \varepsilon_S} \) and \( C_2 = \frac{p \lambda S}{1 - \varepsilon_S} \frac{\pi'(n)n}{\pi} \) where all values are at their non-stochastic steady-state levels and \( \varepsilon_S \) is the elasticity of \( V \) with respect to \( S \).
We focus on a number of special cases. Suppose first that (1) marginal cost is independent of sales, \( p_t = c_t \), and (2) all shocks are permanent: \( \theta = 1 \). Then \( x = 0 \) and \( n \) fluctuates randomly about its mean:

\[
\tilde{n}_t = \frac{s}{\delta} (\tilde{s}_t - \tilde{s}_{t-1}) = \frac{s y}{\delta} \eta_t.
\]

This case reproduces Mankiw’s [32] result. Mankiw finds that with fixed prices and interest rates durables purchases should follow an ARIMA(0,1,1) in response to changes in permanent income in which the lagged MA term is \( 1 - \Delta \). Here the lagged MA term is one, but we are looking only at the number of agents who purchase the durable rather than the total value of the durables that agents purchase. If, as in Mankiw, the innovation \( \eta \) reflects a shock to permanent income, then this shock leads to a permanent increase in \( S \), so that durables purchases \( nS \) follow an ARIMA(0,1,1).

Now suppose that \( \pi' > 0 \), but that shocks are still permanent. In this case (6.4) shows that the number of purchases follows an AR(1) in response to a positive shock:

\[
\tilde{n}_t = \frac{s x}{\delta} \tilde{n}_{t-1} + \frac{s y}{\delta} \eta_t.
\]

It can easily be shown that the value of purchases \( \tilde{q} \) is the sum of an AR(1) and a random walk. Purchases overshoot their long-run level. They rise with the shock and fall gradually back to a new higher level associated with a larger value of \( S \). This is essentially what Caballero [14] found when he fit a high-order MA process to durable consumption.

The intuition for this dynamic in the present case is straightforward. When a positive demand shock hits there would be a spike in purchases if price did not adjust. This spike, however, causes price to rise, which creates an incentive for agents to delay purchases in order to take advantage of lower prices in the future. This delay smooths out the response to the shock. The greater \( \pi' > 0 \), the greater is the price response and the greater is the period over which the shock is spread. Note that \( x \) governs both the impact of the shock through \( y \) and the rate of convergence; persistent responses are associated with muted impact effects.

It is also interesting to consider temporary shocks. If a shock is temporary, then the adjustment trigger \( \kappa \) must eventually return to the same long run position. Therefore, a positive innovation \( \eta \), which leads to above normal purchases in the short run as \( \kappa \) rises, must also eventually lead to below normal purchases at some future date when \( \kappa \) falls. In some sense the increased demand is stolen from the future and a boom-bust cycle ensues. The position of the \( \kappa \) relative to its long run average captures such concepts as “pent up demand” or “exhausted demand.” 17

This analysis reinforces the point that ruling out echo effects does not eliminate all distributional dynamics. The dynamics associated with the cutoff remain. In this sense the model is a complementary to Caballero [15]. Caballero explains the slow adjustment of

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17 Chrysler includes a proxy for pent up demand in its forecasting equations (see [27]).
durable consumption to shocks by the distributional dynamics within the (S,s) band. Here the focus is on the movements in the bands themselves. 18

6.1. Relationship to the representative agent model

We have seen in the case of permanent shocks and flat marginal cost that our simple (S,s) model behaves a lot like the representative agent model without adjustment costs employed by Mankiw [32]. This is for essentially the same reasons discussed by Caplin and Spulber [20]: a few agents making large adjustments may look very similar to many agents making small adjustments, or one large agent making small adjustments.

In principle, the analysis in this section makes it possible to extend this comparison to the case with endogenous prices. This comparison, however, is beyond the scope of this paper. Even if the similarities survive, there are two advantages to our formulation. First, since our (S,s) model is more closely related to the kinds of replacement decisions that agents actually make, it should be more easy to parameterize and test our formulation than it would be to parameterize the representative agent model. In fact, the best way to parameterize a representative agent model may be to match it to a correctly parameterized version of our model. Second, our formulation differentiates between the number and size of purchases. These follow very different processes. Eq. (5.1) shows that purchase size is dependent on \( p_t \lambda_t \), whereas Eq. (6.3) shows that the number of purchases is more closely related to the quasi-first difference \( p_t \lambda_t - \beta p_{t+1} \lambda_{t+1} \). As first argued by Bar-Ilan and Blinder [5], the purchase size follows a random walk if \( p_t \lambda_t \) follows a random walk. As we have seen the number of purchases follows an ARIMA(0,1,1) in this case. In addition to adding a dimension upon which we can parameterize and test the model, it is almost certain that the data on the number of purchases is more accurate than the data on the size of purchases. It is easier to count boxes than it is to measure the quality of the goods inside them.

7. Durable goods monopoly

So far we have considered a competitive market. It is easy to extend the linearized model to the case of a monopolist who chooses price subject to a per period cost of production of \( c_t q_t^\vee \) per unit. For simplicity we fix the size of purchases at \( S = 1 \) and assume that \( c \) and \( \lambda \) follow stationary AR(1)’s.

We look for a perfect Bayesian equilibrium to the game between the monopolist and the consumers. 19 There exists an equilibrium in which the consumer’s cutoff and the

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18 This linearized version of the model shares many similarities with Parker [34]. Both models consider a distribution of consumers holding different amounts of a durable good and emphasize the timing of the decision to purchase durable goods. Both solve for the dynamics of a cutoff rule. Both treat the value of a new durable as exogenous. Parker, however, focuses on the consequences of fluctuations in the density of durable goods holdings. Such fluctuations are ruled out here by the assumption that there are no echo effects and that the density of holdings is log uniform. Parker’s model is non-linear, so he solves a perfect foresight version using computational methods. The uniformity assumption will allow us to solve for the stochastic implications of a variety of shocks in closed form.

19 This model is a stochastic version of Sobel and Takahashi [42] with some elements of Kahn [29].
monopolists pricing strategy take the following form:
\[
\tilde{s}_t = x \tilde{s}_{t-1} + y \tilde{c}_t + z \tilde{\lambda}_t, \tag{7.1}
\]
\[
\tilde{p}_t = m \tilde{s}_{t-1} + w \tilde{c}_t + g \tilde{\lambda}_t. \tag{7.2}
\]
The monopolist maximizes:
\[
J(\tilde{s}_t, \tilde{c}_t, \tilde{\lambda}_t) = \max_{p_t} (p_t - c_t q_t) q_t + \beta E_t J(\tilde{s}_{t+1}, \tilde{c}_{t+1}, \tilde{\lambda}_{t+1}).
\]
The first-order condition for this problem is
\[
q_t + (p_t - (\gamma + 1)c_t q_t^\gamma) \frac{dq_t}{dp_t} + \beta E_t J_1(\tilde{s}_{t+1}, \tilde{c}_{t+1}, \tilde{\lambda}_{t+1}) \frac{d\tilde{s}_{t+1}}{dp_t} = 0. \tag{7.3}
\]
Calculating the derivatives: \(20\)
\[
\frac{dq_t}{dp_t} = \mu \frac{d\tilde{s}_t}{dp_t}; \quad \frac{d\tilde{s}_{t+1}}{dp_t} = x \frac{d\tilde{s}_t}{dp_t}; \quad \frac{d\tilde{s}_t}{dp_t} = -\frac{p \lambda S}{1 - \beta mp \lambda S};
\]
and
\[
J_1(\tilde{s}_{t+1}, \tilde{c}_{t+1}, \tilde{\lambda}_{t+1}) = -\mu \left( p_{t+1} - (\gamma + 1)c_{t+1} q_{t+1}^{\gamma} \right).
\]
Making these substitutions and log linearizing (7.3): \(21\)
\[
(C_1 + \gamma C_2) \tilde{q}_t + (C_1 - C_3) \tilde{p}_t + C_2 \tilde{c}_t = x \beta E_t \left[ C_2 \tilde{c}_{t+1} + \gamma C_2 \tilde{q}_{t+1} - C_3 \tilde{p}_{t+1} \right]. \tag{7.4}
\]
This equation replaces the cost curve in the competitive model. We can solve for the optimal strategies, (7.1) and (7.2), by matching coefficients in (6.1) and (7.4).

This formulation allows us to study markup dynamics. \(22\) With constant costs (\(\gamma = 0\), \(C_2 = 0\) and \(C_1 < C_3\)). In this case, positive demand shocks (reductions in \(\lambda\)) increase the markup of price over marginal cost as the monopolist attempts to take advantage of the greater abundance of buyers. Positive cost shocks (increases in \(c\)) reduce markups. In both cases the monopolist would like to raise price by more but intertemporal substitution in consumers prevents this. The ability to time purchases smooths the price process. Note that monopoly power provides another explanation for the persistence in durable goods demand. In response to a positive demand shock, the monopolist raises its price. Consumers delay purchases, spreading demand over several periods.

\(20\) The only derivative that is not straightforward is \(d\tilde{s}_t/dp_t\). This is an out of equilibrium thought experiment: how will the consumer react to a one-shot deviation in the monopolist’s strategy. To make this calculation totally differentiate equation (6.1) under the assumption that the monopolist plays its equilibrium strategy (7.2) in period \(t+1\). Note also that \(dp_t/dp_t = 1/p_t\).

\(21\) The coefficients are: \(C_1 = (1 - \beta mp \lambda S)q\), \(C_2 = \lambda S \mu (\gamma + 1)c q^\gamma\), and \(C_3 = \lambda S \mu p\).

\(22\) The source of monopoly power in this example is the assumption of discrete time, which makes goods today and goods tomorrow imperfect substitutes. As the period length falls, the Coase conjecture holds and we approach the competitive solution.
Increasing costs ($\gamma > 0$) make the response of the markup to shocks ambiguous, since demand and cost considerations mix. A positive demand shock increases sales. This causes costs to rise and leads the monopolist to raise the price of the durable. The scope for price increases, however, is limited by consumers’ intertemporal substitution.

The model yields conditions under which the markup is countercyclical. If costs are sufficiently increasing or if cost shocks are of sufficient amplitude and positively correlated with demand shocks, then increases in demand will be associated with reductions in the markup.\(^{23}\) Rotemberg and Woodford [39] provide some indirect evidence for this mechanism. They argue that marginal costs increase by more than price during booms, so that real marginal cost is procyclical.

8. Evaluation of the approximation

The accuracy of the approximation depends on the time between purchases. Next we show that if there is enough time then the distribution is arbitrarily flat in the neighborhood of the purchase trigger. We then simulate the model to evaluate its performance for realistic purchase periods.

8.1. Theory

We now present two lemmas that provide some justification for ruling out echo effects. The first lemma presents conditions under which heterogeneity in depreciation smooths the distribution of holdings so that it is eventually log uniform, ignoring, for the moment the fact that purchases truncate the distribution at “little $s$.” The second lemma presents conditions under which this truncation preserves uniformity.

**Lemma 2.** Suppose that on average $\bar{n}$ agents make purchases each period, that the number of agents making purchases is bounded above, and that they purchase a durable that lies in $[\bar{S}, \bar{S}]$, where $\bar{S} > 0$ and $\bar{S} < \infty$. Suppose further that each period durables depreciate according to $K_{it} = \xi_{it}K_{it-1}$ where $\xi$ is a random shock that is independent across consumers, independent across time, and diffuse. In addition, we assume that the mean of $\xi$ is $1 - \Delta$ and that $\xi \in [a, b]$ where $a > 0$ and $b < 1$. Then given $\varepsilon > 0$, there exists $s(\varepsilon) \in (0, S)$ such that the density of log holdings on $(0, s)$ is within $\varepsilon$ of $-\bar{n}/\ln(1 - \Delta)$ according to the variation norm.

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\(^{23}\)This mechanism is suggested by Carlton [21]. There are many other theories of countercyclical markups. Phelps and Winter [35], Bils [10], and Parker [34] provide reasons why demand might be more elastic during booms. Rotemberg and Saloner [37] and Rotemberg and Woodford [38] argue that the incentive to deviate from collusive arrangements is greater during booms. Sobel [41] shows that countercyclical markups may arise from periodic sales to the low valuation consumers. In Chavalier and Scharfstein [22] credit market imperfections lead firms to increase margins when cash flow is low. The theory in this paper is most closely related to the Keynesian view that countercyclical markups arise from price stickiness. Here it is intertemporal substitution by consumers rather than nominal rigidity which prevents the monopolist from adjusting prices.
Lemma 2 says that given enough time heterogeneity in depreciation will smooth the density of holdings to the point that it is arbitrarily close to log uniform. The intuition behind Lemma 2 is simple. Consider water flowing through a pipe. The average flow at any point in the pipe must be the same as the average flow into the pipe. If there is some mixing of the distribution of water within the pipe and if the pipe is sufficiently long then eventually the flow will be fairly constant even if the inflow tends to fluctuate.\footnote{Lemma 2 is related to Theorem 1 in Caballero and Engel \cite{16}. Caballero and Engel show that if agents follow one-sided (S,s) rules and there is a non-stationary idiosyncratic shock to their position between the bands, then the cross-sectional distribution converges to the uniform distribution. Lemma 1 states that this convergence can occur between adjustment if the adjustment period is long enough.}

Lemma 2 ignores the effect that truncating the distribution at $s$ has on the shape of the distribution. If we truncate the density from Lemma 2 at $s$ then it will be log uniform to the right of $s$. Now suppose that in period $t$, we have a distribution that is log uniform to the right of $s$ and zero to the left. In period $t+1$, after depreciation, the distribution will be log uniform to the right of $bs$, zero to the left of $as$ and continuously increasing between these two points. Lemma 3 states that if $s_{t+1} > bs_t$ then successive truncations preserve uniformity.

**Lemma 3.** Suppose that the distribution of log holdings in period $t$ is uniform for $k > \ln s_t$. Suppose $K_{it} = \xi_{it} K_{it-1}$ where $\xi$ is distributed with mean $1 - \Delta$ on $[a, b]$ where $a > 0$ and $b < 1$. Suppose that $\ln s_t - \ln b < \ln s_{t+1}$. Then the distribution of log holdings in period $t+1$ is uniform for $k > \ln s_{t+1}$.

Together Lemmas 2 and 3 show when the uniformity approximation is a good one. It is accurate when there is enough time between purchases for idiosyncratic depreciation to smooth the distribution of holdings and when the purchase trigger does not fall so far as to cut the density of holdings at a region that is not uniform.

### 8.2. Simulations

The ideal test of the approximation would be to compare the approximate model to the exact model and evaluate the difference. It is precisely because we cannot solve the exact model, however, that we need to develop the approximation. We therefore adopt the following strategy. We simulate the market under the assumption that agents behave as if the linear model of Section 6 were true, and then evaluate how well the approximation holds in this setting. This strategy is similar to that of Krussel and Smith \cite{30} or to what Sargent \cite{40} calls a self-confirming equilibrium. We evaluate the model on two margins: does the value of a new durable remain constant over the cycle and does the model accurately forecast next period’s price? To test the latter, we ask how much better price expectations would be if agents took into consideration aspects of the distribution of holdings in addition to the lower bound $\xi_t$.

We parameterize the model to match the U.S. automobile market as closely as possible. The key relationship is Eq. (5.2):

$$V - p_t S_t \lambda_t = \kappa_t + \beta(V - p_t S_t \lambda_t).$$

(8.1)
Here we assume that the utility of holding the durable is $u(K) = \ln K = k$ and that $c$ and $\lambda$ are stationary, so that $V$ is constant. We begin by discretizing the space of utilities to the set of positive integers. We normalize the steady-state value of $\kappa = \ln s$ to 201 (so that agents adjust at 200). As the average new car is held about 5 years, and since a search of the National Association of Automobile Dealers web site indicates that 5 year old cars sell for anywhere between 33% and 60% and the price of new cars, we set $\ln S$ equal to twice $\ln s$ or 400. To simplify matters, we will keep $S$ fixed in the simulation. Allowing $S$ to vary with price actually leads to more mixing of the distribution and improves the accuracy of the approximation.

We take the period of the model to be a quarter. We therefore set the discount rate $\beta = .99$. We specify a depreciation process (described in detail below) on $\{1, 2, \ldots, 400\}$. Given the depreciation process, we can calculate $V$ by calculating the present value of holding a durable of size 400 and noting that whenever the durable depreciates to 201 the remaining value is simply $201/(1 - \beta)$. Given $V$ and $\kappa$, Eq. (8.1) gives us the steady-state value of $pS\lambda$. We choose $pS$ to match the average price of new cars, which is approximately $25,000. This yields $\lambda = 8.62$.

As the cost shock and the marginal utility shock have similar effects, we simulate the effect of a cost shock. We set the standard deviation of the cost shock equal to 1% of the price of the durable. Since the elasticity of demand and persistence of cost shock are both potentially very important parameters, we report results for a range of values for these parameters.

One aspect of the parameterization deserves some discussion is the depreciation process. We initially assumed that each period each car depreciated $d$ utility units, where $d$ is an i.i.d. draw from a uniform distribution with mean $\bar{d}$ and width $w$. We then attempted to select these parameters so that the mean and standard deviation of the steady-state time between purchases matched data from the automobile market, which are approximately 5.1 and 3.1 years respectively. It turned out, however, that it was difficult to get such a long holding time and high variance with i.i.d. depreciation. The reason was that, given positive depreciation, an increase in the variance of depreciation tends to both increase the average rate of depreciation and reduce the holding time. To increase the variance without increasing the drift, we assume that cars with value 400 retained their value of 400 with probability $r$. Choosing $r = .92$, $\bar{d} = 22.5$, and $w = 25$, generates a mean holding time of 20.7 quarters and a standard deviation of 12.1. Note that this amendment to the model only changes the calculation of $V$. All other aspects remain the same. The results were insensitive to alternative parameterizations that yielded similar means and variances.

We start the market with the steady-state distribution of holdings. We then generate a sequence of cost shocks. We assume that agents use Eq. (6.2) to forecast next period’s price conditional on the current value of the cost shock and the current purchase trigger. We then search for the price that cleared the market. We simulated the model over 10,000 periods.

We begin with the accuracy of the price expectations. We ask how much better agents would predict prices if they used information on the distribution of holdings in addition

---

Regressions of price on expected price and \( \varphi_s \) (standard errors in parentheses)

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( \theta = 0 )</th>
<th>( \theta = .9 )</th>
<th>( \theta = 0 )</th>
<th>( \theta = .9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p^e )</td>
<td>( \varphi_s )</td>
<td>( p^e )</td>
<td>( \varphi_s )</td>
</tr>
<tr>
<td>.1</td>
<td>.999 (.012)</td>
<td>479 (121)</td>
<td>1.000 (.006)</td>
<td>172 (281)</td>
</tr>
<tr>
<td>.5</td>
<td>.999 (.008)</td>
<td>1109 (199)</td>
<td>1.001 (.005)</td>
<td>878 (374)</td>
</tr>
<tr>
<td>1</td>
<td>1.000 (.007)</td>
<td>1579 (199)</td>
<td>1.002 (.005)</td>
<td>1445 (440)</td>
</tr>
<tr>
<td>2</td>
<td>1.000 (.007)</td>
<td>2228 (333)</td>
<td>1.003 (.005)</td>
<td>2248 (530)</td>
</tr>
<tr>
<td>10</td>
<td>1.003 (.006)</td>
<td>4679 (620)</td>
<td>1.004 (.004)</td>
<td>5443 (860)</td>
</tr>
</tbody>
</table>

The results of the estimation are presented in Table 1. Each row corresponds to simulations with a different value for the elasticity of the supply curve, \( \varepsilon = (n/p)dp/dn \). The first two columns give coefficients from simulations in which the cost shock is white noise, \( \theta = 0 \), whereas the second two columns correspond to persistent cost shocks, \( \theta = .9 \). A few general observations are immediately apparent. The coefficient on the expected price is very precisely estimated and very close to unity in all of the simulations. Note that \( p^e \) is calculated using the equations from the model. It is therefore significant that we find \( x_1 \approx 1 \). If the approximation is bad there is no reason that Eq. (6.2) should be an unbiased predictor of the future price. The precision of the estimates is, of course, due to the large number of observations. Even if we limit the regression to 100 observations (25 years), however, we usually end up with coefficients within 1% of unity and standard errors less than 4%. The coefficient on the density is relatively poorly estimated given the number of observations. If we limit the regression to 100 observations, this coefficient is often insignificant and sometimes even the wrong sign.

Some of the coefficients on the distributional parameter are quite large. \( x_2 \) tends to be especially large when \( \varepsilon \) is large. Does this imply that when \( \varepsilon \) is large fluctuations in the
distribution of holdings have a large effect on price? The answer is no. The situations in which $\varphi_s$ is large are precisely the situations in which fluctuations in $\varphi_s$ are small. The reason is that when $\varepsilon$ is large, fluctuations in $\varphi_s$ lead to large fluctuations in price. This response smooths out the distribution and ensures that fluctuations in $\varphi_s$ are small. Table 2 presents the standard deviation of price, the standard deviation of $\varphi_s$, and the ratio of the predicted effect of a one the standard deviation increase in $\varphi_s$ to the standard deviation of price. We see that the standard deviation of the distributional parameter is lowest when its effect is greatest. Interestingly, the standard deviation of price also falls with $\varepsilon$, since intertemporal substitution in demand smooths price.

Table 3 shows that the incremental predictive power of the distributional parameter is weak. It compares the $R^2$ from a regression of $p$ on $p^e$ with that from a regression of $p$ on $p^e$ and $\varphi_s$. The expected price accounts for the vast majority of the variation. We conclude that the distributional parameter adds very little to agents’ forecasts.

We now evaluate the constancy of $V$. To do this we select 10 dates from the equilibrium path of the simulation in which $\varepsilon = 1$. We calculate $V$ at each of these dates as follows. At each date $t$, we begin with the state $\{c_t, \kappa_t, \varphi_t\}$ and generate 100 sequences of the cost shock for $\{t + 1, \ldots\}$. For each sequence we calculate the actual value of a new car purchased at date $t$. We then average over these sequences to calculate $V_t$. With 10 draws of $V_t$ from the equilibrium path we then estimate the standard deviation of $V$. As expected, we find that the importance of fluctuations in $V$ depend on the persistence of the cost shock. In the case, that the cost shock is white noise, we find that fluctuations in $V$ are an order of magnitude smaller than fluctuations in the price of the durable. The standard deviation of the former

Table 2
A comparison of the relative volatilities of $p$ and $\varphi_s$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>s.d.(p)</th>
<th>s.d.($\varphi_s$) $\times 10^7$</th>
<th>$\varphi_s \times$ s.d.($\varphi_s$)/s.d.(p)</th>
<th>s.d.(p)</th>
<th>s.d.($\varphi_s$) $\times 10^6$</th>
<th>$\varphi_s \times$ s.d.($\varphi_s$)/s.d.(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>88.65</td>
<td>2954</td>
<td>.00163</td>
<td>499</td>
<td>453</td>
<td>.00015</td>
</tr>
<tr>
<td>.5</td>
<td>55.28</td>
<td>892</td>
<td>.00179</td>
<td>423</td>
<td>263</td>
<td>.00055</td>
</tr>
<tr>
<td>1</td>
<td>44.31</td>
<td>513</td>
<td>.00183</td>
<td>374</td>
<td>190</td>
<td>.00074</td>
</tr>
<tr>
<td>2</td>
<td>31.73</td>
<td>291</td>
<td>.00186</td>
<td>318</td>
<td>130</td>
<td>.00092</td>
</tr>
<tr>
<td>10</td>
<td>17.61</td>
<td>75</td>
<td>.00198</td>
<td>185</td>
<td>44</td>
<td>.00127</td>
</tr>
</tbody>
</table>

Table 3
$R^2$ from regressions of $p$ on $p^e$ with and without $\varphi_s$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\theta = 0$ Without $\varphi_s$</th>
<th>$\theta = 0$ With $\varphi_s$</th>
<th>$\theta = .9$ Without $\varphi_s$</th>
<th>$\theta = .9$ With $\varphi_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.5699</td>
<td>.5712</td>
<td>.8792</td>
<td>.8792</td>
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<tr>
<td>.5</td>
<td>.7623</td>
<td>.7638</td>
<td>.9110</td>
<td>.9111</td>
</tr>
<tr>
<td>1</td>
<td>.8139</td>
<td>.8153</td>
<td>.9228</td>
<td>.9230</td>
</tr>
<tr>
<td>2</td>
<td>.8513</td>
<td>.8527</td>
<td>.9327</td>
<td>.9330</td>
</tr>
<tr>
<td>10</td>
<td>.9014</td>
<td>.9025</td>
<td>.9487</td>
<td>.9490</td>
</tr>
</tbody>
</table>
is 4.0, whereas that of the latter is 44.5. Even in the case that $\theta = .9$, fluctuations price are more important than fluctuations in $V$. The ratio of the standard deviation of $V$ to the standard deviation of price is $2 (71/372)$. Most of this increase in this ratio from the $\theta = 0$ case comes from the persistence of the cost shock. Even after 20 periods (5 years) 10% of the cost shock still remains.

We conclude that the approximation performs remarkably well in these simulations.

9. Conclusion

We have constructed a simple model of the dynamics of a market which respects the discreteness and heterogeneity of individual behavior. The model distinguishes between the dynamics of the optimal size of purchases and the equilibrium number of consumers who make purchases. We have used this model to address a number of issues including the time-series behavior of durables and the behavior of markups. We have shown that if prices rise with demand then purchases will exhibit a prolonged response to wealth shocks. We have also shown that markups are only pro-cyclical if cost shocks are important or marginal cost rises significantly with demand.

Acknowledgments

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Appendix

Proof of Lemma 1. We begin with the differentiability of $\bar{V}$. Consider a consumer with a durable at date $t$ that lies in the neighborhood of $\bar{S}_t$. The agent follows an optimal policy characterized by the functions $s(\omega_t)$ and $S(\omega_t)$. Consider any realization of the depreciation process and of the aggregate state of the market. The optimal policy specifies that the first purchase will take place at some date $T > t$. Now consider a small increase in the initial capital stock. If the timing of the first purchase is not unchanged then the resulting change in the value function is simply the sum of the change in utility up until the first purchase: $\sum_{s=t}^{T-1} \beta^{s-t} K_{x-1}^s$. If the timing of the first purchase does change, the agent must have initially been indifferent between purchasing at $T$ and $T + 1$. Given the continuity of the depreciation, however, these points are measure zero. Integrating over simple paths

$$V_1 = E_t \sum_{s=t}^{T-1} \beta^{s-t} K_{x-1}^s,$$
where it is understood that $T$, the time of the next adjustment, is a random variable. The differentiability of $V_t$ is proved in a similar manner. Given that $V_t$ is twice differentiable, the signs of the derivatives follow immediately from the first- and second-order conditions of optimal choice. This proves (1).

Suppose now that $p_t$ can be written as $\hat{p}_t \tilde{\lambda}_t$ where the trend $\hat{p}_t = \rho_p \eta_{pt} \tilde{p}_{t-1}$ and $\hat{p}_t$ follows a stationary process. We will prove this conjecture below. Let $K^*_t$ denote the optimal holding of the durable if there are no adjustment costs, $p_t = \hat{p}_t$, $\lambda_t = \tilde{\lambda}_t$, and the consumer believes that both $\rho$ and $\lambda$ are expected to grow at their trend growth rates with no shocks to either trend or cycle. $K^*_t$ satisfies the following first-order condition:

$$K^*_{t+1} = \tilde{p}_t \tilde{\lambda}_t (1 - (1 - \Delta) \beta \rho_p \lambda)$$

(\text{A.1})

Here the marginal utility of holding the durable is set equal to a version of Jorgenson's user cost. This proves (4).

Since the actual evolution of $\tilde{p}_t \tilde{\lambda}_t$ is

$$\tilde{p}_t \tilde{\lambda}_t = \rho_p \rho_{\lambda} \eta_{pt} \eta_{\lambda t} \tilde{p}_{t-1} \tilde{\lambda}_{t-1},$$

this implies

$$K^*_t = \rho_p \rho_{\lambda} \eta_{pt} \eta_{\lambda t} K^*_{t-1}. \quad \text{(A.2)}$$

Now consider the optimization problem (3.2), dividing both sides by $K^*_t$, using (A.2) to replace $K^*_t$ with $K^*_s$, and using (A.1) to replace $K^*_{t+1}$ with $\tilde{p}_t \tilde{\lambda}_t$:

$$V_t(K, \tilde{\omega}_t) = \max_{\{T, S_T\}} E_t \left\{ \frac{1}{x} \left( \frac{K_{t+1}}{K^*_t} \right)^x + \sum_{s=t+1}^{T-t} \beta^{s-t} \left( \rho_p \rho_{\lambda} \right) \frac{S_{s-t}}{K^*_t} \right\}$$

$$\times \left( \prod_{r=t+1}^{T} \eta_{pr} \eta_{\lambda r} \right)^{\frac{1}{x}} \left( \frac{K_{s+1}}{K^*_s} \right)^x + \beta^T \left( \rho_p \rho_{\lambda} \right) \frac{S_T}{K^*_T} \right\}.$$  

Since $K_t$ and $K^*$ enter only as the ratio $K_t / K^*$, we can write $V_t(K_t, \tilde{\omega}_t) / K_t^{x-1}$ as $F_t(K_t / K^*_t, \tilde{\omega}_t)$. Moreover, since all of the remaining stochastic variables are cyclical variables that are independent of the current trend state by Assumption 2, we can write $F_t(K_t / K^*_t, \tilde{\omega}_t)$ as $v_t(K_t / K^*_t)$. Substitution into (3.3) establishes (5).

To prove (2), note that the first-order condition for optimal adjustment implies that

$$V'_{t}((\tilde{S}_t, \tilde{\omega}_t)) = \tilde{p}_t \tilde{\lambda}_t.$$  

(3.3)

Since $V_t(K_t, \tilde{\omega}_t) / K_t^{x-1} = F_t(K_t / K^*_t)$, we have $V'_{t}((\tilde{S}_t, \tilde{\omega}_t)) = F'_{t}((\tilde{S}_t / K^*_t) K_t^{x-1})$. Substituting into (A.3) yields

$$F'_{t}((\tilde{S}_t / K^*_t) K_t^{x-1}) = \tilde{p}_t \tilde{\lambda}_t.$$
or
\[ F'(\tilde{S}_t/K^*_t) = \frac{1}{\overline{\Phi}}. \]

Hence \( \tilde{S}_t \) is proportionate to \( K^*_t \). The proportionality of \( \tilde{s}_t \) follows from (3.3) and the homogeneity of \( V \). This proves (2).

We now return to our conjecture concerning the evolution of the price process. Suppose that \( \tilde{p} = \pi(\tilde{n})K^{*\gamma}\tilde{c} \). Let \( \rho_K \) denote the growth rate of \( K^* \). This definition implies:
\[ \rho_p = \rho_K \rho_c. \]

As \( \tilde{p} \) is proportionate to \( K^*^{\gamma-1} \):
\[ \rho_p \rho_\lambda = \rho_K^{\gamma-1}. \]

Eliminating \( \rho_K \):
\[ \rho_p = \frac{\rho_c^{1-\gamma}}{\rho_\lambda^{1-\gamma}} \rho_{\lambda}. \]

Hence \( \tilde{p} \) grows at a constant rate. A similar argument shows
\[ \eta_{pt} = \frac{\rho_c^{1-\gamma} \eta_{\lambda t}}{\rho_{\lambda}^{1-\gamma}}. \]

Given \( \tilde{p} \) we can show that
\[ \tilde{p}_t = \frac{p_t}{p_t} = \frac{\pi(n_t)}{\pi(\tilde{n})} \tilde{S}_t \tilde{c}_t \]
is stationary. This completes proof. \( \square \)

**Proof of Proposition 1.** Define \( h(\tilde{\omega}) = f(\psi(\tilde{\omega})) \). Since \( f \) is monotonically decreasing in \( \psi \), we can recover \( \psi \) from \( h \). In terms of \( h \), Eqs. (5.2) and (5.3) become
\[
\begin{align*}
\hat{h}(\hat{\omega}_t) &= f \left( \zeta \overline{\pi} \left( \mu [g(\hat{\omega}_t) - \kappa_{t-1} + \delta + \ln \hat{e}_t] \right) \hat{S}_t (f^{-1}(h(\hat{\omega}_t))^{\gamma} \hat{e}_t) \right), \\
\hat{h}(\hat{\omega}_t) &= u(g(\hat{\omega}_t)) + \beta E \hat{e}_t h(g(\hat{\omega}_t), \hat{e}_{t+1}, \hat{e}_{t+1}).
\end{align*}
\]

We need to find \( h \) and \( g \) such that these hold for all \( \hat{\omega} \).

Define the mapping \( T: \hat{h} \rightarrow h \) as the solution to the pair of equations:
\[
\begin{align*}
\hat{h}(\hat{\omega}_t) &= f \left( \zeta \overline{\pi} \left( \mu [g(\hat{\omega}_t) - \kappa_{t-1} + \delta + \ln \hat{e}_t] \right) \hat{S}_t (f^{-1}(h(\hat{\omega}_t))^{\gamma} \hat{e}_t) \right), \quad (A.4) \\
\hat{h}(\hat{\omega}_t) &= u(g(\hat{\omega}_t)) + \beta E \hat{e}_t h(g(\hat{\omega}_t), \hat{e}_{t+1}, \hat{e}_{t+1}). \quad (A.5)
\end{align*}
\]

The set of \( h \) that satisfy (A.4) for a given \( g, \kappa, \hat{e}, \) and \( \hat{e} \) are increasing in \( g \). Assuming that \( \hat{h}_k \geq 0 \), it is easy to show that the set of \( h \) that satisfy (A.5) for a given \( g, \kappa, \hat{e}, \) and \( \hat{e} \) are decreasing in \( g \). Hence, given \( \hat{h}_k \geq 0 \), \( T \) is well defined.
Given $\hat{h}_k \geq 0$, it is easy to show graphically that an increase in $\kappa$ leads to a rise in $h$. Therefore $T$ preserves the monotonicity of $\hat{h}$. We therefore look for a fixed point in the space of bounded continuous non-decreasing functions with the sup norm. 27

Given $\beta E_t \hat{\epsilon}_{t+1} < 1$, it is easy to show graphically that an increase in $\hat{h}$ to $\hat{h} + m$ increases $h$ by less than $\beta m E_t \hat{\epsilon}_{t+1}$. Also $\hat{h}_1 > \hat{h}_2$ implies $h_1 > h_2$. The mapping therefore satisfies Blackwell’s conditions for a contraction mapping, and there exists a unique fixed point $h^*$ by the Contraction Mapping Theorem. From $h^*$ we can deduce the equilibrium $\psi$ and $g$. This establishes that there exists a unique equilibrium in the market. □

**Proof of Lemma 2.** The statement follows from the properties of the diffusion equation.

**Proof of Lemma 3.** Immediate.

**References**


27 Since our utility function is not naturally bounded. We impose a bound a some very high level of $\hat{K}$ that is unlikely to be reached in equilibrium.


